Maximum-norm stability of the finite element Stokes projection

V. Girault* R.H. Nochetto† R. Scott‡

March 30, 2004

Abstract

We prove stability of the finite element Stokes projection in the product space $W^{1,\infty}(\Omega) \times L^\infty(\Omega)$, i.e., the maximum norm of the discrete velocity gradient and discrete pressure are bounded by the sum of the corresponding exact counterparts, independently of the mesh-size. The proof relies on weighted L^2 estimates for regularized Green's functions associated with the Stokes problem and on a weighted inf-sup condition. The domain is a polygon or polyhedron with a Lipschitz-continuous boundary, satisfying suitable sufficient conditions on the inner angles of its boundary, so that the exact solution is bounded in $W^{1,\infty}(\Omega) \times L^\infty(\Omega)$. The triangulation is shape-regular and quasi-uniform. The finite element spaces satisfy a super-approximation property, which is shown to be valid for commonly used stable finite element spaces.

Résumé: Nous démontrons la stabilité dans $W^{1,\infty}(\Omega) \times L^\infty(\Omega)$ de l'approximation par éléments finis du problème de Stokes, i.e., la norme du maximum du gradient de la vitesse et celle de la pression, calculés par des méthodes d'éléments finis usuelles pour discrétiser le problème de Stokes, sont bornées indépendemment du pas de la discrétisation. La démonstration est basée sur des estimations à poids dans L^2 pour des fonctions de Green associées au problème de Stokes et sur une condition inf-sup à poids. Le domaine est un polygone ou un polyèdre à frontière Lipschitz dont les angles intérieurs satisfont des conditions suffisantes convenables pour assurer que la solution exacte est aussi bornée dans $W^{1,\infty}(\Omega) \times L^\infty(\Omega)$. La triangulation est uniformément régulière.

Keywords: Stokes problem, finite element method, Green's function, duality argument, weighted error estimates, weighted inf-sup condition, local interpolation

AMS Classification: 65N15, 65N30, 76D07

0 Introduction

This article is devoted to the proof of estimates, in the maximum norm, for the gradient of the velocity of the discrete Stokes projection and its associated pressure in a variety of finite-element spaces. We consider a polygonal or polyhedral domain Ω, in two or three dimensions d, a given velocity vector $\boldsymbol{u}$ in $H_0^1(\Omega)^d$, with zero divergence, and a pressure p in $L_0^2(\Omega)$, i.e. with zero mean value. Then we consider a triangulation $\mathcal{T}_h$ of $\overline{\Omega}$, where h is the global mesh-size, a pair of finite-element spaces on $\mathcal{T}_h$, namely $X_h \subset H_0^1(\Omega)^d$ and

*Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, 75252 Paris cedex 05, France.

†Department of Mathematics and Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742–4015, USA. (Partially supported by NSF grant DMS-0204670.)

‡Department of Mathematics and the Computation Institute, University of Chicago, Chicago, Illinois 60637–1581, USA.

$M_h \subset L^2_0(\Omega)$, with appropriate approximation properties and stable in the sense that it satisfies a uniform discrete inf-sup condition. We define $\boldsymbol{u}_h \in X_h$ and $p_h \in M_h$, solution of:

$$\int_\Omega \nabla \boldsymbol{u}_h : \nabla \boldsymbol{v}_h \, d\boldsymbol{x} - \int_\Omega p_h \mathrm{div}\, \boldsymbol{v}_h \, d\boldsymbol{x} = \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v}_h \, d\boldsymbol{x} - \int_\Omega p \,\mathrm{div}\, \boldsymbol{v}_h \, d\boldsymbol{x} \quad \forall \boldsymbol{v}_h \in X_h \,, \tag{0.1}$$

$$\int_\Omega q_h \mathrm{div}\, \boldsymbol{u}_h \, d\boldsymbol{x} = 0 \quad \forall q_h \in M_h \,. \tag{0.2}$$

Under suitable sufficient restrictions on the angles of the domain and on the triangulation, we shall prove that if the velocity $\boldsymbol{u}$ belongs to $W^{1,\infty}(\Omega)^d$ and the pressure p belongs to $L^\infty(\Omega)$, then there exists a constant C independent of h, $\boldsymbol{u}$ and p, such that

$$\|\nabla \boldsymbol{u}_h\|_{L^\infty(\Omega)} + \|p_h\|_{L^\infty(\Omega)} \le C \left(\|\nabla \boldsymbol{u}\|_{L^\infty(\Omega)} + \|p\|_{L^\infty(\Omega)} \right) \,. \tag{0.3}$$

This result has many important applications. For instance, it is crucial for extending to Navier-Stokes free surface flows the numerical analysis done by Saavedra & Scott [33] for the discrete Laplace equation with a free surface. Another application is the numerical analysis of finite-element schemes for highly non-linear flows such as non-Newtonian flows. Analyzing such flows often requires a $W^{1,\infty}$ bound for the exact velocity; thus the numerical analysis of their finite-element schemes requires a similar bound for the discrete velocity. One example is the numerical analysis of finite-element schemes for a grade-two fluid flow in three dimensions. In two dimensions, (0.3) is not required, cf. Girault & Scott [21], but this is exceptional and (0.3) is essential in three dimensions.

It is well-known that

$$\|\nabla \boldsymbol{u}_h\|_{L^2(\Omega)} \le \|\nabla \boldsymbol{u}\|_{L^2(\Omega)} + \|p\|_{L^2(\Omega)} \,, \tag{0.4}$$

and

$$\|p_h\|_{L^2(\Omega)} \le \frac{1}{\beta^\star} \left(\|\nabla \boldsymbol{u}\|_{L^2(\Omega)} + \|p\|_{L^2(\Omega)} \right) , \tag{0.5}$$

where $\beta^\star$ is the constant of the uniform discrete inf-sup condition:

$$\sup_{\boldsymbol{v}_h \in X_h} \frac{\int_\Omega q_h \mathrm{div}\, \boldsymbol{v}_h \, d\boldsymbol{x}}{\|\nabla \boldsymbol{v}_h\|_{L^2(\Omega)}} \ge \beta^\star \|q_h\|_{L^2(\Omega)} \quad \forall q_h \in M_h \,. \tag{0.6}$$

Therefore, by interpolating between (0.3) and (0.4) or (0.5), we obtain for any number $r > 2$:

$$\|\nabla \boldsymbol{u}_h\|_{L^r(\Omega)} + \|p_h\|_{L^r(\Omega)} \le C_r \left(\|\nabla \boldsymbol{u}\|_{L^r(\Omega)} + \|p\|_{L^r(\Omega)} \right) \,, \tag{0.7}$$

with a constant C_r that depends on r, but not on h, $\boldsymbol{u}$ and p.

0.1 Some background

The result we present here is based essentially on the proof of two results: maximum norm estimates for (the gradient of) finite-element discretizations of the Laplace equation due to Rannacher & Scott [32] and Brenner & Scott [7], and a family of weighted estimates for the inverse of the divergence due to Durán & Muschietti [15]. The reader will find in the recent work by Schatz [34], page 878, a good summary of the history of maximum norm estimates for the Laplace equation.

In 1988, Durán, Nochetto & Wang [16] addressed the discrete Stokes problem in two dimensions by means of weighted norms with the weight function introduced by Natterer [28]:

$$\sigma(\boldsymbol{x}) = \left(|\boldsymbol{x} - \boldsymbol{x}_0|^2 + (\kappa\, h)^2 \right)^{1/2} \,, \tag{0.8}$$

where $\boldsymbol{x}_0$ is a point close to that where the maximum is attained and $\kappa > 1$ is a well-chosen parameter independent of h. But their estimate was not uniform: their constant C had the factor $|\log \mathrm{h}|^{1/2}$. This difficulty in proving $W^{1,\infty}$-stability is not due to the degree of the finite elements, as experienced by Ciarlet & Raviart [11], Scott [35] or Nitsche [30, 31] when dealing with the Laplace equation. It is caused by the presence of the discrete pressure in the discrete equations, even for estimating the velocity. Indeed, in the absence of weights, the discrete pressure can be eliminated by using test functions with discrete zero divergence: this is how (0.4) is derived. But in the presence of weights multiplying the test functions, the pressure cannot be eliminated since the discrete divergence of the product is no longer zero. Unfortunately, the weighted inf-sup condition for handling this pressure term has a constant with a logarithmic factor, and this accounts for the factor found in [16]. But in 2001, Durán & Muschietti [15] proved what amounts to a uniform weighted inf-sup condition with the weight σ^α for all exponents α with $-d/2 < \alpha < d/2$, d being the dimension, and exhibiting the factor $|\log \mathrm{h}|$ in the critical case where $|\alpha| = d/2$. Their proof uses fundamentally a theorem of Stein [37] establishing a sharp estimate for a general singular integral with the weight $|\boldsymbol{x}|^\alpha$ for $-d/2 < \alpha < d/2$.

Recently, in a preprint, Chen [9] presented maximum norm estimates on a domain with a smooth boundary, but without specifying the behavior of the finite-element functions near the boundary. This work is based on interior estimates (away from the boundary) for the Stokes problem by Arnold & Liu [4], and recent pointwise estimates by Schatz for the Laplace equation [34]. The approach of [34] has been extended by Demlow [14] to mixed methods for solving scalar elliptic problems on smooth domains.

In this article, we shall adapt the analysis of [7] to the Stokes problem, removing the logarithmic factor by working with the weight $\sigma^{\mu/2}$, where

$$\mu = d + \lambda\ ,\ \ 0 < \lambda < 1\,, \tag{0.9}$$

λ is a well-chosen parameter and d is the dimension. We shall transform the contribution of the discrete pressure in such a way that the inf-sup condition is only applied in a non-critical case. Let us describe briefly the main steps in the proof.

0.2 Synopsis of the proof

The *first step*, which is standard, consists in reducing the estimate for $\boldsymbol{u}_h$ in $W^{1,\infty}$ into an error estimate for a regularized Green's function first in $W^{1,1}$, and next in H^1 with a weight. For this, we fix an element of the matrix $\nabla\, \boldsymbol{u}_h$, say $\frac{\partial \boldsymbol{u}_{h,i}}{\partial x_j}$, we choose a suitable point $\boldsymbol{x}_0$ located in the element T (triangle or tetrahedron) where $|\frac{\partial \boldsymbol{u}_{h,i}}{\partial x_j}|$ is maximum, and an approximate mollifier δ_M supported by T, satisfying:

$$\int_\Omega \delta_M d\boldsymbol{x} = 1\,, \tag{0.10}$$

$$\|\frac{\partial \boldsymbol{u}_{h,i}}{\partial x_j}\|_{L^\infty(\Omega)} = \left| \int_\Omega \delta_M\, \frac{\partial \boldsymbol{u}_{h,i}}{\partial x_j}\, d\boldsymbol{x} \right|\,. \tag{0.11}$$

Next, we define the regularized Green's function by: $(\boldsymbol{G}, Q) \in H^1_0(\Omega)^d \times L^2_0(\Omega)$, solution of

$$-\Delta\, \boldsymbol{G} + \nabla\, Q = -\frac{\partial}{\partial x_j}(\delta_M \boldsymbol{e}_i)\,, \tag{0.12}$$

$$\mathrm{div}\, \boldsymbol{G} = 0\,, \tag{0.13}$$

where $\boldsymbol{e}_i$ is the ith unit canonical vector, and we define its Stokes projection $(\boldsymbol{G}_h, Q_h) \in X_h \times M_h$ by:

$$\int_\Omega \nabla\, \boldsymbol{G}_h : \nabla\, \boldsymbol{v}_h\, d\boldsymbol{x} - \int_\Omega Q_h \mathrm{div}\, \boldsymbol{v}_h\, d\boldsymbol{x} = \int_\Omega \nabla\, \boldsymbol{G} : \nabla\, \boldsymbol{v}_h\, d\boldsymbol{x} - \int_\Omega Q\, \mathrm{div}\, \boldsymbol{v}_h\, d\boldsymbol{x} \quad \forall \boldsymbol{v}_h \in X_h\,, \tag{0.14}$$

$$\int_\Omega q_h \mathrm{div}\, \boldsymbol{G}_h \, d\boldsymbol{x} = 0 \quad \forall q_h \in M_h \,. \tag{0.15}$$

Then, we can show that

$$\int_\Omega \delta_M \frac{\partial \boldsymbol{u}_{h,i}}{\partial x_j}\, d\boldsymbol{x} = \int_\Omega \delta_M \frac{\partial \boldsymbol{u}_i}{\partial x_j}\, d\boldsymbol{x} - \int_\Omega \nabla \boldsymbol{u} : \nabla(\boldsymbol{G} - \boldsymbol{G}_h)\, d\boldsymbol{x} + \int_\Omega p\, \mathrm{div}(\boldsymbol{G} - \boldsymbol{G}_h)\, d\boldsymbol{x} \,, \tag{0.16}$$

and combined with (0.11), this implies indeed that the problem reduces to a uniform estimate for $\|\nabla(\boldsymbol{G} - \boldsymbol{G}_h)\|_{L^1(\Omega)}$. Finally, using Cauchy-Schwarz's inequality, we write:

$$\|\nabla(\boldsymbol{G} - \boldsymbol{G}_h)\|_{L^1(\Omega)} \le \left(\int_\Omega \sigma^\mu |\nabla(\boldsymbol{G} - \boldsymbol{G}_h)|^2 d\boldsymbol{x}\right)^{1/2} \left(\int_\Omega \sigma^{-\mu} d\boldsymbol{x}\right)^{1/2} . \tag{0.17}$$

As we can easily prove that for $0 < \lambda < 1$,

$$\int_\Omega \sigma^{-\mu} d\boldsymbol{x} \le \frac{C}{\lambda} (\kappa h)^{-\lambda} \,, \tag{0.18}$$

with a constant C independent of h and λ, this reduces now the problem to establishing the weighted error estimate for $\boldsymbol{G}_h$:

$$\|\sigma^{\mu/2} \nabla(\boldsymbol{G} - \boldsymbol{G}_h)\|_{L^2(\Omega)} \le C h^{\lambda/2} \,. \tag{0.19}$$

But since (0.14) is a variational equation, the only straightforward way for introducing a weight into it is by inserting the weight into the test function. For this, we interpolate $\boldsymbol{G}$ with an interpolation operator P_h that *preserves the discrete divergence* [22], we define the auxiliary function $\boldsymbol{\psi}$ by

$$\boldsymbol{\psi} = \sigma^\mu (P_h(\boldsymbol{G}) - \boldsymbol{G}_h) \,, \tag{0.20}$$

and we use $\bar{P}_h(\boldsymbol{\psi})$ as test function, where $\bar{P}_h$ is a simplified version of P_h that takes advantage of the continuity of $\boldsymbol{\psi}$. This yields the following identity:

$$\begin{aligned} \int_\Omega \sigma^\mu |\nabla(\boldsymbol{G} - \boldsymbol{G}_h)|^2\, d\boldsymbol{x} = & \int_\Omega \nabla(\boldsymbol{G} - \boldsymbol{G}_h) : \nabla\left[(\boldsymbol{G} - P_h(\boldsymbol{G}))\sigma^\mu\right]\, d\boldsymbol{x} \\ & + \int_\Omega \nabla(\boldsymbol{G} - \boldsymbol{G}_h) : \nabla(\boldsymbol{\psi} - \bar{P}_h(\boldsymbol{\psi}))\, d\boldsymbol{x} \\ & - \int_\Omega (\nabla(\boldsymbol{G} - \boldsymbol{G}_h)(\boldsymbol{G} - \boldsymbol{G}_h)) \cdot \nabla \sigma^\mu\, d\boldsymbol{x} \\ & + \int_\Omega (Q - Q_h) \mathrm{div}\,(\bar{P}_h(\boldsymbol{\psi}))\, d\boldsymbol{x} \,. \end{aligned} \tag{0.21}$$

All the subsequent steps are devoted to estimating the terms in the right-hand side of (0.21).

In view of the first term, we must derive a weighted estimate for the interpolation error $\nabla(\boldsymbol{G} - P_h(\boldsymbol{G}))$. This is the object of the *second step*, that establishes the weighted bounds:

$$\|\sigma^{\mu/2} \nabla_2 \boldsymbol{G}\|_{L^2(\Omega)} + \|\sigma^{\mu/2} \nabla\, Q\|_{L^2(\Omega)} \le C\, \kappa^{\mu/2} h^{\lambda/2 - 1} \,, \tag{0.22}$$

where ∇_k denotes the kth-order derivatives tensor. It is essentially based on two arguments: a duality argument for $\boldsymbol{G}$, similar to that used by [32] and [7], and a weighted inf-sup condition for Q, that applies [15] with the non-critical exponent $\alpha = -(\mu/2 - 1)$ utilizing $0 < \lambda < 1$. Let us remark here that a weighted estimate for the interpolation error of P_h also requires that P_h be quasi-local. For this, we refer to [22], where quasi-local interpolation operators are constructed for a large class of finite-elements.

The second term in the right-hand side of (0.21) involves a weighted estimate for $\nabla(\boldsymbol{\psi} - \bar{P}_h(\boldsymbol{\psi}))$. More specifically, we shall prove that

$$\|\sigma^{-\mu/2}\nabla(\boldsymbol{\psi} - \bar{P}_h(\boldsymbol{\psi}))\|_{L^2(\Omega)} \le C\|\sigma^{\mu/2-1}(P_h(\boldsymbol{G}) - \boldsymbol{G}_h)\|_{L^2(\Omega)}\,, \tag{0.23}$$

with a constant C independent of h. Since $\boldsymbol{\psi}$ has the factor σ^μ, we shall see that (0.23) follows mainly from a "super-approximation" result that eliminates the highest-order derivative of $P_h(\boldsymbol{G}) - \boldsymbol{G}_h$ in the right-hand side of the error bound. The *third step* is devoted to establishing this "super-approximation" result for the "mini-element", the Taylor-Hood finite elements and the Bernardi-Raugel element.

The *fourth step* is motivated by the last two terms in the right-hand side of (0.21). On the one hand, the third term has the bound:

$$\int_\Omega (\nabla(\boldsymbol{G} - \boldsymbol{G}_h)(\boldsymbol{G} - \boldsymbol{G}_h)) \cdot \nabla\,\sigma^\mu\,d\boldsymbol{x} \le \mu\|\sigma^{\mu/2}\nabla(\boldsymbol{G} - \boldsymbol{G}_h)\|_{L^2(\Omega)}\|\sigma^{\mu/2-1}(\boldsymbol{G} - \boldsymbol{G}_h)\|_{L^2(\Omega)}\,.$$

On the other hand, the fourth term, i.e. the one involving the pressure, can be reduced essentially to two terms:

$$\int_\Omega (r_h(Q) - Q_h)\sigma^\mu \mathrm{div}(\boldsymbol{G}_h - P_h(\boldsymbol{G}))\,d\boldsymbol{x}\ ,\ \int_\Omega (Q - Q_h)(\boldsymbol{G}_h - P_h(\boldsymbol{G}))\nabla\,\sigma^\mu\,d\boldsymbol{x}\,, \tag{0.24}$$

where $r_h(Q)$ is an interpolant of Q. The first term in (0.24) is simpler because $\boldsymbol{G}_h - P_h(\boldsymbol{G})$ has discrete divergence zero. Thus we can insert an approximation of the product $(r_h(Q) - Q_h)\sigma^\mu$ into this term and we shall prove that

$$\int_\Omega (r_h(Q) - Q_h)\sigma^\mu \mathrm{div}(\boldsymbol{G}_h - P_h(\boldsymbol{G}))\,d\boldsymbol{x} \le C\,h\,\|\sigma^{\mu/2}\nabla(\boldsymbol{G} - \boldsymbol{G}_h)\|_{L^2(\Omega)}\|\sigma^{\mu/2-1}(r_h(Q) - Q_h)\|_{L^2(\Omega)}\,. \tag{0.25}$$

Then the pressure factor in the right-hand side can be estimated by means of the weighted inf-sup condition with non-critical exponent $-(\mu/2 - 1)$, since $0 < \lambda < 1$, and we shall see that the factor h exactly compensates the -1 in the above exponent.

The second term in (0.24) is much more problematic because the obvious factorization, which after simplification gives

$$\int_\Omega (Q - Q_h)(\boldsymbol{G} - \boldsymbol{G}_h) \cdot \nabla\,\sigma^\mu\,d\boldsymbol{x} \le \mu\|\sigma^{\mu/2}(Q - Q_h)\|_{L^2(\Omega)}\|\sigma^{\mu/2-1}(\boldsymbol{G} - \boldsymbol{G}_h)\|_{L^2(\Omega)}\,,$$

is useless as it requires the weighted inf-sup condition with exponent $-\mu/2$, i.e. beyond the admissible range. In order to stay within the non-critical range, we consider the factorization

$$\int_\Omega (Q - Q_h)(\boldsymbol{G} - \boldsymbol{G}_h) \cdot \nabla\,\sigma^\mu\,d\boldsymbol{x} \le \mu\|\sigma^{\mu/2-\varepsilon/2}(Q - Q_h)\|_{L^2(\Omega)}\|\sigma^{\mu/2+\varepsilon/2-1}(\boldsymbol{G} - \boldsymbol{G}_h)\|_{L^2(\Omega)}\,, \tag{0.26}$$

where $\varepsilon = \lambda + \gamma$ for some small $\gamma > 0$. Thus, in view of these two terms, and since λ itself is also small, we are led to find an appropriate bound for

$$\|\sigma^{\mu/2+\varepsilon/2-1}(\boldsymbol{G} - \boldsymbol{G}_h)\|_{L^2(\Omega)}\,, \tag{0.27}$$

for small $\varepsilon \ge 0$. We shall estimate it by means a duality argument that generalizes the argument of [32] and [7] for evaluating (0.27) with $\varepsilon = 0$. We shall prove first that

$$\|\sigma^{\mu/2-1}(\boldsymbol{G} - \boldsymbol{G}_h)\|_{L^2(\Omega)} \le \frac{C_1}{\sqrt{\kappa}}\|\sigma^{\mu/2}\nabla(\boldsymbol{G} - \boldsymbol{G}_h)\|_{L^2(\Omega)} + C_2\kappa^{\mu/2-1/2}h^{\lambda/2}\,, \tag{0.28}$$

and next that

$$\|\sigma^{\mu/2+\varepsilon/2-1}(\boldsymbol{G}-\boldsymbol{G}_h)\|_{L^2(\Omega)} \le C_3 \frac{(\kappa\, h)^{\varepsilon/2}}{\sqrt{\kappa}} (\|\sigma^{\mu/2}\nabla(\boldsymbol{G}-\boldsymbol{G}_h)\|_{L^2(\Omega)} + C_4\kappa^{\mu/2}h^{\lambda/2})\,. \tag{0.29}$$

Observe on the one hand that the factor $h^{\varepsilon/2}$ exactly compensates the $-\varepsilon/2$ in the exponent of the first factor of the right-hand side of (0.26). On the other hand, the parameter κ, that is part of the weight (0.8), appears in the denominator multiplying $\|\sigma^{\mu/2}\nabla(\boldsymbol{G}-\boldsymbol{G}_h)\|_{L^2(\Omega)}$. Hence it will be chosen so that this term is absorbed by the left-hand side of (0.21).

The remainder of the proof assembles all these estimates in such a way that all contributions of $\|\sigma^{\mu/2}\nabla(\boldsymbol{G}-\boldsymbol{G}_h)\|_{L^2(\Omega)}$ in the right-hand side of (0.21) are absorbed by its left-hand side.

We shall see that several steps in this proof restrict the triangulation. Indeed, since σ is a function of the global mesh-size, the proofs of some estimates use a uniformly regular (or quasi-uniform) triangulation. This is also the case in [32] and [7]. But relaxing, even partially, this restriction is not straightforward.

The above duality argument restricts from the start the angles of $\partial\Omega$. Indeed, in view of the Sobolev imbedding

$$W^{2,r}(\Omega) \subset W^{1,\infty}(\Omega)\ ,\ \text{for } r > d\,,$$

the angles must be such that the solution $(\boldsymbol{v}, q)$ of the Stokes problem

$$-\Delta\, \boldsymbol{v} + \nabla\, q = \boldsymbol{f}\ ,\ \operatorname{div} \boldsymbol{v} = 0\ \text{ in } \Omega\ ,\ \boldsymbol{v}|_{\partial\Omega} = \boldsymbol{0}\,, \tag{0.30}$$

belongs to $W^{2,r}(\Omega)^d \times W^{1,r}(\Omega)$ whenever $\boldsymbol{f}$ belongs to $L^r(\Omega)^d$ for some real number $r > d$. In two dimensions, this holds when Ω is convex, and r depends on the largest inner angle of $\partial\Omega$ (see Grisvard [23]). But in three dimensions, convexity is not sufficient (see Dauge [13]): the largest inner dihedral angle of $\partial\Omega$ must be strictly less than $2\pi/3$, the precise value depending on r. This amount of regularity is essentially consistent with requiring that p and the gradient of $\boldsymbol{u}$ be bounded, in the sense that the restriction on the angles is the same. Thus our restrictions on the boundary are best possible consistent with our goal of providing error estimates for the approximation of p and the gradient of $\boldsymbol{u}$ in the maximum norm.

0.3 Regularity results for the Stokes problem

It is worthwhile here to recall some regularity results of the solution of the Stokes problem. It is now well-known that if $\boldsymbol{f}$ belongs to $L^2(\Omega)^d$ and the domain is a convex polygon (cf. Kellog & Osborn [26] or [23]) or polyhedron (cf. [13]), then the solution $(\boldsymbol{v}, q)$ of (0.30) belongs to $H^2(\Omega)^d \times H^1(\Omega)$, with continuous dependence on $\boldsymbol{f}$. Of course when Ω is convex, we obtain by interpolation for $0 \le s \le 1$, that $(\boldsymbol{v}, q)$ belongs to $H^{s+1}(\Omega)^d \times H^s(\Omega)$, with continuous dependence on $\boldsymbol{f}$, whenever $\boldsymbol{f}$ belongs to $H^{s-1}(\Omega)^d$. But for small s, the restrictions on the angles can be substantially relaxed. Indeed, without restriction on the angles of $\partial\Omega$, the following theorems hold; the first one can be found in [23] and the second one in [13].

Theorem 0.1 *Let Ω be a polygon with a Lipschitz-continuous boundary in two dimensions. If $\boldsymbol{f}$ belongs to $L^r(\Omega)^2$ for some r with $1 < r \le 4/3$, then the solution $(\boldsymbol{v}, q)$ of (0.30) belongs to $W^{2,r}(\Omega)^2 \times W^{1,r}(\Omega)$ with continuous dependence on $\boldsymbol{f}$.*

Theorem 0.2 *Let Ω be a polyhedron with a Lipschitz-continuous boundary in three dimensions. If $\boldsymbol{f}$ belongs to $H^{s-1}(\Omega)^3$ for some s with $0 \le s < 1/2$, then the solution $(\boldsymbol{v}, q)$ of (0.30) belongs to $H^{s+1}(\Omega)^3 \times H^s(\Omega)$ with continuous dependence on $\boldsymbol{f}$.*

The result for the borderline case $s = 1/2$ is due to Dauge & Costabel and can be found in Girault & Lions [19]:

Theorem 0.3 *Let Ω be a polyhedron with a Lipschitz-continuous boundary in three dimensions. If $\boldsymbol{f}$ belongs to $L^{3/2}(\Omega)^3$, then the solution $(\boldsymbol{v}, q)$ of (0.30) belongs to $H^{3/2}(\Omega)^3 \times H^{1/2}(\Omega)$ with continuous dependence on $\boldsymbol{f}$.*

Finally, there are several results for handling the Stokes problem with non-zero divergence. We shall use the following one due to Amrouche & Girault [2] (see also [26] in two dimensions):

Theorem 0.4 *Let Ω be a Lipschitz-continuous domain of $\mathbb{R}^d$. For each g in $H_0^1(\Omega)$ satisfying $\int_\Omega g\, d\boldsymbol{x} = 0$, there exists a unique $\boldsymbol{v}$ in $H_0^2(\Omega)^d$ such that*

$$\operatorname{div} \boldsymbol{v} = g \ , \ \ \|\boldsymbol{v}\|_{H^2(\Omega)} \le C\, \|g\|_{H^1(\Omega)} \, . \tag{0.31}$$

0.4 Notation

We shall use the following notation; for the sake of simplicity, we define them in three dimensions. Let (k_1, k_2, k_3) denote a triple of non-negative integers, set $|k| = k_1 + k_2 + k_3$ and define the partial derivative ∂^k by

$$\partial^k v = \frac{\partial^{|k|} v}{\partial x_1^{k_1} \partial x_2^{k_2} \partial x_3^{k_3}} \, .$$

Then, for any non-negative integer m and number $r \ge 1$, recall the classical Sobolev space (cf. Adams [1] or Nečas [29])

$$W^{m,r}(\Omega) = \{ v \in L^r(\Omega) \, ; \, \partial^k v \in L^r(\Omega) \ \forall |k| \le m \} \, ,$$

equipped with the seminorm

$$|v|_{W^{m,r}(\Omega)} = \left[\sum_{|k|=m} \int_\Omega |\partial^k v|^r \, d\boldsymbol{x} \right]^{1/r} \, ,$$

and norm (for which it is a Banach space)

$$\|v\|_{W^{m,r}(\Omega)} = \left[\sum_{0 \le k \le m} |v|^r_{W^{k,r}(\Omega)} \right]^{1/r} \, ,$$

with the usual extension when $r = \infty$. The reader can refer to Lions & Magenes [27] and [23] for extensions of this definition to non-integral values of m. When $r = 2$, this space is the Hilbert space $H^m(\Omega)$. The definitions of these spaces are extended straightforwardly to vectors, with the same notation, but with the following modification for the norms in the non-Hilbert case. Let $\boldsymbol{u} = (\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3)$; then we set

$$\|\boldsymbol{u}\|_{L^r(\Omega)} = \left[\int_\Omega |\boldsymbol{u}(\boldsymbol{x})|^r \, d\boldsymbol{x} \right]^{1/r} \, ,$$

where $|\cdot|$ denotes the Euclidean vector norm for vectors or the Frobenius norm for tensors.

Let $\mathcal{D}(\Omega)$ denote the set of indefinitely differentiable functions with compact support in Ω. For functions that vanish on the boundary, we define

$$H_0^1(\Omega) = \{ v \in H^1(\Omega) \, ; \, v|_{\partial\Omega} = 0 \} \, ,$$

and recall Poincaré's inequality: there exists a constant C such that

$$\|v\|_{L^2(\Omega)} \le C \operatorname{diam}(\Omega) |v|_{H^1(\Omega)} \quad \forall v \in H_0^1(\Omega) \, . \tag{0.32}$$

Owing to (0.32), we use the seminorm $|\cdot|_{H^1(\Omega)}$ as a norm on $H_0^1(\Omega)$.

For $R > 0$, we denote by $B(\boldsymbol{x}, R)$ the ball in $\mathbb{R}^d$ with center $\boldsymbol{x}$ and radius R.

We shall also use the standard spaces for incompressible fluids:

$$V = \{\boldsymbol{v} \in H_0^1(\Omega)^3\,;\, \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega\}\,,$$

$$V^\perp = \{\boldsymbol{v} \in H_0^1(\Omega)^3\,;\, \int_\Omega \nabla\, \boldsymbol{v} : \nabla\, \boldsymbol{w} = 0 \quad \forall \boldsymbol{w} \in V\,\}\,,$$

$$L_0^2(\Omega) = \{q \in L^2(\Omega)\,;\, \int_\Omega q\, d\boldsymbol{x} = 0\}\,.$$

1 Reduction to weighted estimates

Let Ω be a Lipschitz-continuous domain in $\mathbb{R}^d$ ($d = 2$ or 3), with a polygonal or polyhedral boundary $\partial\Omega$. We denote by $\delta_1 \geq 0$ the usual mollifier in $\mathcal{D}(\mathbb{R}^d)$ such that supp $(\delta_1) \subset B_1 = B(0,1)$ and $\int_{\mathbb{R}^d} \delta_1(\boldsymbol{x})\, d\boldsymbol{x} = 1$. Then for any point $\boldsymbol{x}_0 \in \Omega$ and real number $\varrho_0 > 0$ such that the ball $B(\boldsymbol{x}_0, \varrho_0)$ is contained in Ω, we define the mollifier δ by:

$$\delta = \frac{1}{\varrho_0^d} \delta_1 \left(\frac{\boldsymbol{x} - \boldsymbol{x}_0}{\varrho_0}\right)\,. \tag{1.1}$$

Let

$$c_1 = \|\delta_1\|_{L^\infty(\mathbb{R}^d)}\,;$$

then

$$\|\delta\|_{L^\infty(\mathbb{R}^d)} = \frac{c_1}{\varrho_0^d}\,. \tag{1.2}$$

Let $\mathcal{T}_h$ be a shape-regular (also called non-degenerate) simplicial family of triangulations of $\overline{\Omega}$ (cf. Ciarlet [10]): there exists a constant ζ, independent of h and T, such that

$$\zeta_T := \frac{h_T}{\rho_T} \leq \zeta \quad \forall T \in \mathcal{T}_h\,, \tag{1.3}$$

where h_T is the diameter of T and ρ_T is the diameter of the sphere B inscribed in T; elements T are assumed to be closed. We denote the center of B by $\boldsymbol{x}_0$ and its radius by $\varrho_0 := \rho_T/2$; i.e. $B = B(\boldsymbol{x}_0, \varrho_0)$.

Our first lemma associates an approximate mollifier with the maximum of a discrete function. This construction is sketched in [32]; we give the proof for the reader's convenience (see [16] for an alternate approach). For a fixed integer $\ell \geq 0$, let $\mathcal{P}_\ell$ be a space of polynomials in d variables of degree at most ℓ, let φ_h be a polynomial of $\mathcal{P}_\ell$ in each T (without interelement continuity requirements), let $\boldsymbol{x}_M$ be a point of $\overline{\Omega}$ where $|\varphi_h(\boldsymbol{x})|$ attains its maximum, let T be an element containing $\boldsymbol{x}_M$ and let B be the sphere associated above with T.

Lemma 1.1 *With the above notation, there exists a smooth function δ_M supported by B such that*

$$\int_\Omega \delta_M d\boldsymbol{x} = 1\,, \tag{1.4}$$

$$\|\varphi_h\|_{L^\infty(\Omega)} = \left|\int_B \delta_M\, \varphi_h d\boldsymbol{x}\right|\,, \tag{1.5}$$

and for any number t with $1 < t \leq \infty$, there exists a constant C_t, depending only on ζ, d, t and the dimension of $\mathcal{P}_\ell$, such that

$$\|\delta_M\|_{L^t(B)} \leq \frac{C_t}{\varrho_T^{d(1-1/t)}}\,. \tag{1.6}$$

Proof. Let δ be defined by (1.1) and let $p_M \in \mathcal{P}_\ell$ be the solution of:

$$\int_B \delta\, p_M\, v\, d\boldsymbol{x} = v(\boldsymbol{x}_M) \quad \forall v \in \mathcal{P}_\ell\,. \tag{1.7}$$

This problem is a square system of linear equations with the dimension of $\mathcal{P}_\ell$; its matrix is symmetric and as δ is positive in the interior of B, it is positive definite. Therefore it has a unique solution and we choose

$$\delta_M = \delta\, p_M\,. \tag{1.8}$$

Then (1.4) and (1.5) follow immediately from (1.7), and (1.6) is easily proven by scaling arguments. ∎

Now, we proceed as sketched in the Introduction. We apply Lemma 1.1 to $\frac{\partial \boldsymbol{u}_{h,i}}{\partial x_j}$, i.e. $\mathcal{P}_\ell$ is the polynomial space of the first derivatives of $\boldsymbol{u}_h$ in each T. This gives the existence of a point $\boldsymbol{x}_0$ and a corresponding function δ_M satisfying (1.4)–(1.6). With Green's formula, we have:

$$\|\frac{\partial \boldsymbol{u}_{h,i}}{\partial x_j}\|_{L^\infty(\Omega)} = \left| \int_\Omega \frac{\partial}{\partial x_j}(\delta_M\, \boldsymbol{e}_i) \cdot \boldsymbol{u}_h\, d\boldsymbol{x} \right|\,. \tag{1.9}$$

Then defining the regularized Green's function $(\boldsymbol{G}, Q)$ by (0.12), (0.13) and its Stokes projection $(\boldsymbol{G}_h, Q_h)$ by (0.14), (0.15), we have the following lemma.

Lemma 1.2 *Let $\boldsymbol{u}$ be given in $W^{1,\infty}(\Omega)^d \cap V$ and p in $L^\infty(\Omega) \cap L^2_0(\Omega)$ and let the pair $(\boldsymbol{u}_h, p_h)$ be the solution of (0.1), (0.2). Then*

$$\|\frac{\partial \boldsymbol{u}_{h,i}}{\partial x_j}\|_{L^\infty(\Omega)} \le \|\frac{\partial \boldsymbol{u}_i}{\partial x_j}\|_{L^\infty(\Omega)} + (\|\nabla \boldsymbol{u}\|_{L^\infty(\Omega)} + \sqrt{d}\|p\|_{L^\infty(\Omega)})\|\nabla(\boldsymbol{G}_h - \boldsymbol{G})\|_{L^1(\Omega)}\,. \tag{1.10}$$

Proof. Taking $\boldsymbol{u}_h$ as test function in (0.14) and using (0.2), we can write

$$-\int_\Omega \frac{\partial}{\partial x_j}(\delta_M\, \boldsymbol{e}_i) \cdot \boldsymbol{u}_h\, d\boldsymbol{x} = \int_\Omega \nabla \boldsymbol{G}_h : \nabla \boldsymbol{u}_h\, d\boldsymbol{x}\,.$$

Then taking $\boldsymbol{G}_h$ as test function in (0.1) and using (0.13), we obtain

$$-\int_\Omega \frac{\partial}{\partial x_j}(\delta_M\, \boldsymbol{e}_i) \cdot \boldsymbol{u}_h\, d\boldsymbol{x} = \int_\Omega \nabla \boldsymbol{u} : \nabla(\boldsymbol{G}_h - \boldsymbol{G})\, d\boldsymbol{x} + \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{G}\, d\boldsymbol{x} - \int_\Omega p \operatorname{div}(\boldsymbol{G}_h - \boldsymbol{G})\, d\boldsymbol{x}\,.$$

Finally, multiplying (0.12) by $\boldsymbol{u}$, this becomes

$$-\int_\Omega \frac{\partial}{\partial x_j}(\delta_M\, \boldsymbol{e}_i) \cdot \boldsymbol{u}_h\, d\boldsymbol{x} = \int_\Omega \nabla \boldsymbol{u} : \nabla(\boldsymbol{G}_h - \boldsymbol{G})\, d\boldsymbol{x} + \int_\Omega \frac{\partial \boldsymbol{u}_i}{\partial x_j}\delta_M\, d\boldsymbol{x} - \int_\Omega p \operatorname{div}(\boldsymbol{G}_h - \boldsymbol{G})\, d\boldsymbol{x}\,,$$

and (1.10) follows from the fact that $\|\delta_M\|_{L^1(\Omega)} = 1$. ∎

Next we introduce the notation

$$\theta := \kappa\, h\,, \tag{1.11}$$

with $\kappa > 1$ to be specified later, and recall the weight σ defined by (0.8)

$$\sigma(\boldsymbol{x}) = \left(|\boldsymbol{x} - \boldsymbol{x}_0|^2 + (\kappa\, h)^2\right)^{1/2} = \left(|\boldsymbol{x} - \boldsymbol{x}_0|^2 + \theta^2\right)^{1/2}\,.$$

The following proof of (0.18) gives an explicit bound for its constant C.

Lemma 1.3 *Let $\mu = d + \lambda$, $d = 2, 3$. For all $\lambda > 0$, we have:*

$$\int_\Omega \sigma(\boldsymbol{x})^{-\mu} d\boldsymbol{x} \leq C_{\lambda,d} \frac{1}{\theta^\lambda} \quad \textit{where} \quad C_{\lambda,d} = \int_{I\!R^d} (1 + |\boldsymbol{x}|^2)^{-d-\lambda} d\boldsymbol{x} \leq 2(d-1) \left(\frac{\pi}{\lambda}\right). \tag{1.12}$$

Proof. By changing $\boldsymbol{x}$ to

$$\boldsymbol{y} = \frac{\boldsymbol{x} - \boldsymbol{x}_0}{\theta}$$

and passing to spherical coordinates, we obtain

$$\int_\Omega \sigma(\boldsymbol{x})^{-\mu} d\boldsymbol{x} \leq \int_{I\!R^d} \sigma(\boldsymbol{x})^{-\mu} d\boldsymbol{x} = 2(d-1)\pi \frac{1}{\theta^\lambda} \int_0^\infty \frac{r^{d-1}}{(r^2+1)^{\mu/2}} dr\,.$$

Then (1.12) follows from the fact that

$$\int_0^\infty \frac{r^{d-1}}{(r^2+1)^{\mu/2}} dr \leq \frac{1}{\lambda}\,.$$

∎

Remark 1.4 Let us fix once and for all a ball centered at the point $\boldsymbol{x}_0$, with radius R, containing Ω. If $0 < \alpha < d$, we have

$$\int_\Omega \sigma^{-\alpha}(\boldsymbol{x}) d\boldsymbol{x} \leq 2\,\pi \frac{d-1}{d-\alpha} R^{d-\alpha}\,, \tag{1.13}$$

and if $\alpha > 0$, we have for $\theta \leq R$

$$\int_\Omega \sigma^{\alpha}(\boldsymbol{x}) d\boldsymbol{x} \leq 2^{1+\alpha/2}\,\pi \frac{d-1}{d} R^{d+\alpha}\,. \tag{1.14}$$

If $R < \theta$, then (1.14) holds with R replaced by θ, but this case is irrelevant since θ tends to zero with h. ∎

In view of Lemma 1.2, the proof of (0.3) reduces to proving the weighted estimate (0.19):

$$\int_\Omega \sigma^\mu |\nabla(\boldsymbol{G} - \boldsymbol{G}_h)|^2 d\boldsymbol{x} \leq C h^\lambda\,. \tag{1.15}$$

For this, we need to insert the factor σ^μ into the error equation (0.14). As written in the Introduction, we deal with this factor by means of a test function, but since the product $\sigma^\mu \boldsymbol{v}_h$ does not belong to X_h, we must interpolate it. Therefore, we take an interpolation operator $P_h : H_0^1(\Omega)^d \mapsto X_h$ and a simplified version $\bar{P}_h$, both to be specified later, and we define $\boldsymbol{\psi}$ by (0.20), namely

$$\boldsymbol{\psi} = \sigma^\mu (P_h(\boldsymbol{G}) - \boldsymbol{G}_h)\,.$$

We further choose $\boldsymbol{v}_h = \bar{P}_h(\boldsymbol{\psi})$ in (0.14) to get

$$\int_\Omega \nabla(\boldsymbol{G} - \boldsymbol{G}_h) : \nabla\, \bar{P}_h(\boldsymbol{\psi})\, d\boldsymbol{x} = \int_\Omega (Q - Q_h) \mathrm{div}\, \bar{P}_h(\boldsymbol{\psi})\, d\boldsymbol{x}\,. \tag{1.16}$$

Then we write

$$\int_\Omega \sigma^\mu |\nabla(\boldsymbol{G} - \boldsymbol{G}_h)|^2\, d\boldsymbol{x} = \int_\Omega \nabla(\boldsymbol{G} - \boldsymbol{G}_h) : \nabla\left[(\boldsymbol{G} - \boldsymbol{G}_h)\sigma^\mu\right]\, d\boldsymbol{x} - \int_\Omega (\nabla(\boldsymbol{G} - \boldsymbol{G}_h)(\boldsymbol{G} - \boldsymbol{G}_h)) \cdot \nabla\, \sigma^\mu\, d\boldsymbol{x}\,,$$

and we obtain (0.21) by inserting $P_h(\boldsymbol{G})$, $\bar{P}_h(\boldsymbol{\psi})$ and using (1.16).

2 Preliminary results

From now on, we assume (0.9) for some $\lambda > 0$:

$$\mu = d + \lambda\,.$$

We list here some technical results that will be used repeatedly in the sequel. First, we shall need a bound for the derivatives of powers of σ. As

$$\nabla\,\sigma(\boldsymbol{x}) = \frac{1}{\sigma(\boldsymbol{x})}(\boldsymbol{x}-\boldsymbol{x}_0)\,,$$

we have

$$\nabla(\sigma(\boldsymbol{x})^\alpha) = \alpha\,\sigma(\boldsymbol{x})^{\alpha-2}(\boldsymbol{x}-\boldsymbol{x}_0)\,.$$

Therefore

$$|\nabla(\sigma(\boldsymbol{x})^\alpha)| \le \alpha\,\sigma(\boldsymbol{x})^{\alpha-1}\,, \tag{2.1}$$

and similarly, for any positive integer k:

$$|\nabla_k(\sigma(\boldsymbol{x})^\alpha)| \le C_{k,\alpha}\sigma(\boldsymbol{x})^{\alpha-k}\,, \tag{2.2}$$

with a constant $C_{k,\alpha}$ that depends only on α and k. Next, we shall use the following lemmas. Beforehand, recall that a macro-element is a union of elements of $\mathcal{T}_h$ with a connected interior.

Lemma 2.1 *Let T be any element of $\mathcal{T}_h$. For any real number $\alpha > 0$, we have*

$$\frac{\sup_{\boldsymbol{x}\in T}\sigma(\boldsymbol{x})^{-\alpha}}{\inf_{\boldsymbol{x}\in T}\sigma(\boldsymbol{x})^{-\alpha}} = \frac{\sup_{\boldsymbol{x}\in T}\sigma(\boldsymbol{x})^{\alpha}}{\inf_{\boldsymbol{x}\in T}\sigma(\boldsymbol{x})^{\alpha}} < 3^{\alpha/2}\,. \tag{2.3}$$

Similarly, let Δ_T be a macro-element containing at most L elements of $\mathcal{T}_h$, including T. Then

$$\begin{aligned}\frac{\sup_{\boldsymbol{x}\in T}\sigma(\boldsymbol{x})^{-\alpha}}{\inf_{\boldsymbol{x}\in \Delta_T}\sigma(\boldsymbol{x})^{-\alpha}} = \frac{\sup_{\boldsymbol{x}\in \Delta_T}\sigma(\boldsymbol{x})^{\alpha}}{\inf_{\boldsymbol{x}\in T}\sigma(\boldsymbol{x})^{\alpha}} &< (2L^2+1)^{\alpha/2}\,,\\ \frac{\sup_{\boldsymbol{x}\in T}\sigma(\boldsymbol{x})^{\alpha}}{\inf_{\boldsymbol{x}\in \Delta_T}\sigma(\boldsymbol{x})^{\alpha}} &< (2L^2+1)^{\alpha/2}\,.\end{aligned} \tag{2.4}$$

Proof. The equality in (2.3) is clear. To prove the inequality, let $\boldsymbol{x}_m \in T$ be a point where $|\boldsymbol{x}-\boldsymbol{x}_0|$ attains its minimum in T (if $\boldsymbol{x}_0 \in T$, then $\boldsymbol{x}_m = \boldsymbol{x}_0$). Then on the one hand,

$$\sigma(\boldsymbol{x}) \ge \sigma(\boldsymbol{x}_m) = (|\boldsymbol{x}_m-\boldsymbol{x}_0|^2+\theta^2)^{1/2} \quad \forall \boldsymbol{x}\in T\,,$$

and on the other hand, since $\theta \ge h$, for all $\boldsymbol{x}\in T$,

$$\sigma(\boldsymbol{x}) \le (2\,|\boldsymbol{x}-\boldsymbol{x}_m|^2 + 2\,|\boldsymbol{x}_m-\boldsymbol{x}_0|^2+\theta^2)^{1/2} \le (2\,h^2+\theta^2+2\,|\boldsymbol{x}_m-\boldsymbol{x}_0|^2)^{1/2} < 3^{1/2}\sigma(\boldsymbol{x}_m)\,,$$

whence (2.3). The proof of the first part of (2.4) is similar, considering that, for all $\boldsymbol{x}\in\Delta_T$, $|\boldsymbol{x}-\boldsymbol{x}_m| \le L\,h$. Likewise, for proving the second part of (2.4), we choose for $\boldsymbol{x}_m$ a point where $|\boldsymbol{x}-\boldsymbol{x}_0|$ attains its minimum in Δ_T, and we proceed as in the first part. ∎

Lemma 2.2 *In addition to (1.3), assume that the family of triangulation $\mathcal{T}_h$ is uniformly regular (or quasi-uniform), i.e. there exists a constant $\tau > 0$, independent of h, such that*

$$\tau\, h < h_T \le \zeta\, \rho_T \quad \forall T \in \mathcal{T}_h\,. \tag{2.5}$$

Then there exists a constant C that depends only on τ, ζ, d and the dimension of $\mathcal{P}_\ell$, such that

$$\|\sigma^{\mu/2}\nabla\,\delta_M\|_{L^2(\Omega)} \le 2^{\mu/4} C\,\kappa^{\mu/2} h^{\lambda/2-1}\,, \tag{2.6}$$

and

$$\|\sigma^{\mu/2-1}\delta_M\|_{L^2(\Omega)} \le 2^{\mu/4-1/2} C\,\kappa^{\mu/2-1} h^{\lambda/2-1}\,. \tag{2.7}$$

Proof. From the construction of Lemma 1.1, we have

$$\|\delta_M\|_{L^\infty(\Omega)} = \|\delta_M\|_{L^\infty(B)} \le \frac{\hat{c}_1}{\varrho_0^d}\,, \qquad \|\nabla\,\delta_M\|_{L^\infty(\Omega)} = \|\nabla\,\delta_M\|_{L^\infty(B)} \le \frac{\hat{c}_2}{\varrho_0^{d+1}}\,,$$

with constants $\hat{c}_1, \hat{c}_2 > 0$ that depend only on d, ζ and the dimension of $\mathcal{P}_\ell$. Similarly, for any $\alpha > 0$,

$$\|\sigma^\alpha\|_{L^\infty(B)} \le (\varrho_0^2 + \theta^2)^{\alpha/2} < 2^{\alpha/2}\theta^\alpha\,.$$

Hence

$$\|\sigma^{\mu/2}\nabla\,\delta_M\|_{L^2(\Omega)} = \|\sigma^{\mu/2}\nabla\,\delta_M\|_{L^2(B)} < 2^{\mu/4}\hat{c}_3 \left(\frac{\theta^\mu}{\varrho_0^{d+2}}\right)^{1/2},$$

$$\|\sigma^{\mu/2-1}\delta_M\|_{L^2(\Omega)} = \|\sigma^{\mu/2-1}\delta_M\|_{L^2(B)} < 2^{\mu/4-1/2}\hat{c}_4 \left(\frac{\theta^{\mu-2}}{\varrho_0^{d}}\right)^{1/2},$$

with $\hat{c}_3, \hat{c}_4 > 0$ similar to $\hat{c}_1, \hat{c}_2$. Then (2.6) and (2.7) follow from these two inequalities and (2.5). ∎

Finally, the weighted inf-sup condition stated in the following theorem will be a crucial ingredient here. As written in the Introduction, this result is due to Durán & Muschietti [15] and it generalizes a theorem of Stein [37].

Theorem 2.3 *Let $\Omega \subset \mathbb{R}^d$ be Lipschitz-continuous, and let $\alpha \in \mathbb{R}$ satisfy $0 \le |\alpha| < d$. For each $f \in L^2_0(\Omega)$, there exists $\boldsymbol{v} \in H^1_0(\Omega)^d$ such that*

$$\operatorname{div} \boldsymbol{v} = f \text{ in } \Omega\,,$$

$$\|\sigma^{\alpha/2}\nabla\,\boldsymbol{v}\|_{L^2(\Omega)} \le C_\alpha \|\sigma^{\alpha/2} f\|_{L^2(\Omega)}\,, \tag{2.8}$$

where C_α is a constant that is independent of h, κ, f and $\boldsymbol{v}$.

Remark 2.4 In particular, it is proven in [15] that if $\alpha = d$, then

$$C_d = \mathrm{O}(|\log h|)\,.$$

Thus the condition $|\alpha| < d$ is sharp. ∎

The last corollary handles the case where the mean-value of f is not zero.

Corollary 2.5 *We retain the assumptions of Theorem 2.3. For each $f \in L^2(\Omega)$, there exists $\boldsymbol{v} \in H_0^1(\Omega)^d$ such that*

$$\operatorname{div} \boldsymbol{v} = f - \frac{1}{|\Omega|}\int_\Omega f(\boldsymbol{x})d\boldsymbol{x} \text{ in } \Omega\,,$$

$$\|\sigma^{\alpha/2}\nabla\,\boldsymbol{v}\|_{L^2(\Omega)} \le C_\alpha\|\sigma^{\alpha/2}f\|_{L^2(\Omega)}\,, \tag{2.9}$$

where C_α is another constant that is independent of h, κ, f and $\boldsymbol{v}$.

Proof. To simplify, we consider the case where $\alpha < 0$; the proof for $\alpha \ge 0$ is the same. Set

$$m(f) = \frac{1}{|\Omega|}\int_\Omega f(\boldsymbol{x})d\boldsymbol{x}\,. \tag{2.10}$$

By virtue of (1.14) we write:

$$|\int_\Omega f(\boldsymbol{x})d\boldsymbol{x}| \le \|\sigma^{\alpha/2}f\|_{L^2(\Omega)}\|\sigma^{-\alpha/2}\|_{L^2(\Omega)} \le C_1\|\sigma^{\alpha/2}f\|_{L^2(\Omega)}\,.$$

Then (1.13) yields

$$\|\sigma^{\alpha/2}m(f)\|_{L^2(\Omega)} = |m(f)|\|\sigma^{\alpha/2}\|_{L^2(\Omega)} \le C_2\|\sigma^{\alpha/2}f\|_{L^2(\Omega)}\,. \tag{2.11}$$

With (2.8), this implies immediately (2.9). ∎

3 Weighted interpolation errors

From now on, C, C_i, C_t, etc. will denote generic constants, independent of h and κ.

3.1 Weighted regularity results

For establishing (0.22), we require weighted estimates for $\boldsymbol{G}$ and Q. Let us start with an estimate for $\sigma^{\mu/2-1}Q$ in L^2. Unless otherwise mentioned, we assume that $0 < \lambda < 1$, and we shall sharpen this range in the next section.

Proposition 3.1 *Let $\mathcal{T}_h$ satisfy (2.5) and $0 < \lambda < 2$. We have*

$$\|\sigma^{\mu/2-1}Q\|_{L^2(\Omega)} \le C\left(\|\sigma^{\mu/2-1}\nabla\,\boldsymbol{G}\|_{L^2(\Omega)} + \kappa^{\mu/2-1}h^{\lambda/2-1}\right)\,. \tag{3.1}$$

Proof. Set $q = \sigma^{\mu-2}Q$ and apply Corollary 2.5 with $\alpha = 2-\mu$ and q instead of f. Note that since $0 < \lambda < 2$, we have $-d < 2-\mu < 0$. Therefore, there exists $\boldsymbol{v} \in H_0^1(\Omega)^d$ such that $\operatorname{div}\boldsymbol{v} = -q + m(q)$ and

$$\|\sigma^{1-\mu/2}\nabla\,\boldsymbol{v}\|_{L^2(\Omega)} \le C_\alpha\|\sigma^{1-\mu/2}q\|_{L^2(\Omega)}\ ,\ \alpha = 2-\mu\,. \tag{3.2}$$

On the one hand, as $Q \in L_0^2(\Omega)$, we can write:

$$-\int_\Omega Q\operatorname{div}\boldsymbol{v}\,d\boldsymbol{x} = \int_\Omega \sigma^{\mu-2}Q^2 d\boldsymbol{x}\,,$$

and hence

$$\|\sigma^{\mu/2-1}Q\|_{L^2(\Omega)} = -\frac{\int_\Omega Q\operatorname{div}\boldsymbol{v}\,d\boldsymbol{x}}{\|\sigma^{\mu/2-1}Q\|_{L^2(\Omega)}}\,. \tag{3.3}$$

On the other hand, applying (3.2),

$$\int_\Omega \sigma^{\mu-2}Q^2 d\boldsymbol{x} = \int_\Omega \sigma^{2-\mu}(\sigma^{\mu-2}Q)^2 d\boldsymbol{x} = \int_\Omega \sigma^{2-\mu} q^2 d\boldsymbol{x} \geq \frac{1}{C_\alpha^2}\|\sigma^{1-\mu/2}\nabla\,\boldsymbol{v}\|^2_{L^2(\Omega)}\,.$$

When substituted into the denominator of (3.3), we obtain

$$\|\sigma^{\mu/2-1}Q\|_{L^2(\Omega)} \leq C_\alpha \frac{|\int_\Omega Q\,\mathrm{div}\,\boldsymbol{v}\,d\boldsymbol{x}|}{\|\sigma^{1-\mu/2}\nabla\,\boldsymbol{v}\|_{L^2(\Omega)}}\,. \tag{3.4}$$

Finally, multiplying (0.12) by $\boldsymbol{v}$ yields

$$-\int_\Omega Q\,\mathrm{div}\,\boldsymbol{v}\,d\boldsymbol{x} = \int_\Omega \delta_M \frac{\partial}{\partial x_j}\boldsymbol{v}_i d\boldsymbol{x} - \int_\Omega \nabla\,\boldsymbol{G} : \nabla\,\boldsymbol{v}\,d\boldsymbol{x}\,.$$

Thus

$$|\int_\Omega Q\,\mathrm{div}\,\boldsymbol{v}\,d\boldsymbol{x}| \leq \|\sigma^{1-\mu/2}\nabla\,\boldsymbol{v}\|_{L^2(\Omega)}\left(\|\sigma^{\mu/2-1}\nabla\,\boldsymbol{G}\|_{L^2(\Omega)} + \|\sigma^{\mu/2-1}\delta_M\|_{L^2(\Omega)}\right)\,,$$

and (3.1) follows from this inequality, (3.4) and (2.7). ∎

In view of (3.1), we must find a bound for $\sigma^{\mu/2-1}\nabla\,\boldsymbol{G}$.

Proposition 3.2 *Let $\mathcal{T}_h$ satisfy (2.5). Then*

$$\|\sigma^{\mu/2-1}\nabla\,\boldsymbol{G}\|^2_{L^2(\Omega)} \leq \|\sigma^{\mu/2-2}\boldsymbol{G}\|_{L^2(\Omega)}\left(C_1\kappa^{\mu/2}h^{\lambda/2-1} + C_2\|\sigma^{\mu/2-1}\nabla\,\boldsymbol{G}\|_{L^2(\Omega)}\right)\,. \tag{3.5}$$

Proof. We have

$$\|\sigma^{\mu/2-1}\nabla\,\boldsymbol{G}\|^2_{L^2(\Omega)} = \int_\Omega \nabla\,\boldsymbol{G} : \sigma^{\mu-2}\nabla\,\boldsymbol{G}\,d\boldsymbol{x} = \int_\Omega \nabla\,\boldsymbol{G} : \nabla(\sigma^{\mu-2}\boldsymbol{G})d\boldsymbol{x} - \int_\Omega (\nabla\,\boldsymbol{G})\boldsymbol{G}\cdot\nabla(\sigma^{\mu-2})d\boldsymbol{x}\,.$$

Multiplying (0.12) by $\sigma^{\mu-2}\boldsymbol{G}$ and using (0.13), this becomes

$$\|\sigma^{\mu/2-1}\nabla\,\boldsymbol{G}\|^2_{L^2(\Omega)} = -\int_\Omega \sigma^{\mu-2}\boldsymbol{G}_i\frac{\partial}{\partial x_j}\delta_M d\boldsymbol{x} + \int_\Omega Q\,\nabla(\sigma^{\mu-2})\cdot\boldsymbol{G}\,d\boldsymbol{x} - \int_\Omega (\nabla\,\boldsymbol{G})\boldsymbol{G}\cdot\nabla(\sigma^{\mu-2})d\boldsymbol{x}\,.$$

Then, associating the factor $\sigma^{\mu/2-2}$ with $\boldsymbol{G}$ and applying (2.1) with exponent $\mu-2$, we obtain

$$\begin{aligned}\|\sigma^{\mu/2-1}\nabla\,\boldsymbol{G}\|^2_{L^2(\Omega)} \leq{}& \|\sigma^{\mu/2-2}\boldsymbol{G}\|_{L^2(\Omega)}(\|\sigma^{\mu/2}\nabla\,\delta_M\|_{L^2(\Omega)}\\ &+(\mu-2)\|\sigma^{\mu/2-1}Q\|_{L^2(\Omega)} + (\mu-2)\|\sigma^{\mu/2-1}\nabla\,\boldsymbol{G}\|_{L^2(\Omega)})\,.\end{aligned}$$

Hence (3.5) follows by substituting (3.1) and (2.6) into this inequality. ∎

Next, in view of (3.5), we must find a bound for $\sigma^{\mu/2-2}\boldsymbol{G}$. This is achieved by a duality argument as in [32] and [7].

Theorem 3.3 *Assume that Ω is convex and let $\mathcal{T}_h$ satisfy (2.5). For each real number t satisfying*

$$1 < t < \frac{2\,d}{2\,d+\lambda-2}\,, \tag{3.6}$$

there exists a constant C_t such that the following bound holds

$$\|\sigma^{\mu/2-2}\boldsymbol{G}\|_{L^2(\Omega)} \leq C_t\kappa^{d(1-1/t)+\lambda/2-1}h^{\lambda/2-1}\,. \tag{3.7}$$

Proof. Owing to Sobolev's imbedding, $\boldsymbol{G}$ belongs to $L^{2s}(\Omega)^d$ for any real number $s > 1$ when $d = 2$ and $s \le 3$ when $d = 3$; then we can write

$$\|\sigma^{\mu/2-2}\boldsymbol{G}\|_{L^2(\Omega)} \le \|\boldsymbol{G}\|_{L^{2s}(\Omega)} \left(\int_\Omega \sigma^{(\mu-4)s'} d\boldsymbol{x}\right)^{1/2s'},$$

with $1/s + 1/s' = 1$. We want to apply Lemma 1.3 to the above integral. This requires that $(4-\mu)s' > d$. As $0 < \lambda < 1$ and $d \ge 2$, we have both $4 - \mu > 0$ and $\frac{d}{4-\mu} > 1$. Hence this condition is equivalent to

$$s < \frac{d}{2\,d + \lambda - 4}. \tag{3.8}$$

Then Lemma 1.3 implies, with the constant of (1.12):

$$\|\sigma^{\mu/2-2}\boldsymbol{G}\|_{L^2(\Omega)} \le \left(C_{(4-\mu)s'-d,d}\,\theta^{d-(4-\mu)s'}\right)^{1/2s'} \|\boldsymbol{G}\|_{L^{2s}(\Omega)}. \tag{3.9}$$

Now, we proceed by duality. Let $(\boldsymbol{w}, r) \in H^1_0(\Omega)^d \times L^2_0(\Omega)$ be the solution of the Stokes problem:

$$-\Delta\,\boldsymbol{w} + \nabla\,r = |\boldsymbol{G}|^{2s-2}\boldsymbol{G}\ ,\ \operatorname{div}\boldsymbol{w} = 0 \text{ in } \Omega\ ,\ \boldsymbol{w} = \boldsymbol{0} \text{ on } \partial\Omega. \tag{3.10}$$

On the one hand, since $|\boldsymbol{G}|^{2s-2}\boldsymbol{G}$ belongs to $L^{2s/(2s-1)}(\Omega)^d$ and since

$$1 < \frac{2\,s}{2\,s-1} < 2,$$

the convexity of Ω implies that $\boldsymbol{w} \in W^{2,2s/(2s-1)}(\Omega)^d$ with

$$\|\boldsymbol{w}\|_{W^{2,2s/(2s-1)}(\Omega)} \le C_1 \||\boldsymbol{G}|^{2s-1}\|_{L^{2s/(2s-1)}(\Omega)} = C_1 \left(\int_\Omega |\boldsymbol{G}|^{2s} d\boldsymbol{x}\right)^{(2s-1)/2s}. \tag{3.11}$$

On the other hand, multiplying the first equation in (3.10) by $\boldsymbol{G}$, multiplying (0.12) by $\boldsymbol{w}$, using (0.13) and the second equation in (3.10), we derive

$$\int_\Omega |\boldsymbol{G}|^{2s} d\boldsymbol{x} = \int_\Omega \nabla\,\boldsymbol{w} : \nabla\,\boldsymbol{G}\,d\boldsymbol{x} = \int_\Omega \delta_M \frac{\partial}{\partial x_j}\boldsymbol{w}_i d\boldsymbol{x}. \tag{3.12}$$

Hence,

$$\|\boldsymbol{G}\|^{2s}_{L^{2s}(\Omega)} \le \|\delta_M\|_{L^t(\Omega)} \|\nabla\,\boldsymbol{w}\|_{L^{t'}(\Omega)}\ ,\qquad \frac{1}{t} + \frac{1}{t'} = 1, \tag{3.13}$$

for any $t' > 1$ such that $W^{2,2s/(2s-1)}(\Omega) \subset W^{1,t'}(\Omega)$. This imbedding holds if

$$\frac{1}{t'} = 1 - \frac{1}{2\,s} - \frac{1}{d} \qquad \text{i.e.} \quad \frac{1}{t} = \frac{1}{2\,s} + \frac{1}{d}.$$

As $s > 1$ and $d \ge 2$, this condition gives $t > 1$ and in view of (3.8), it gives (3.6). Now, substituting (1.6) and (3.11) into (3.13) and simplifying, we obtain

$$\|\boldsymbol{G}\|_{L^{2s}(\Omega)} \le \frac{C_2}{\varrho_T^{d(1-1/t)}},$$

and (3.7) follows by substituting this inequality into (3.9) and using (2.5). ∎

Corollary 3.4 *With the assumptions and notation of Theorem 3.3, we have*

$$\|\sigma^{\mu/2-1}\nabla\,\boldsymbol{G}\|_{L^2(\Omega)} + \|\sigma^{\mu/2-1}Q\|_{L^2(\Omega)} \le C_t\kappa^{(\lambda-1+d(3/2-1/t))/2}h^{\lambda/2-1}\,. \tag{3.14}$$

Proof. Applying Young's inequality to the second term in (3.5), we obtain with the same constants:

$$\|\sigma^{\mu/2-1}\nabla\,\boldsymbol{G}\|^2_{L^2(\Omega)} \le C_2^2\|\sigma^{\mu/2-2}\boldsymbol{G}\|^2_{L^2(\Omega)} + 2\,C_1\kappa^{\mu/2}h^{\lambda/2-1}\|\sigma^{\mu/2-2}\boldsymbol{G}\|_{L^2(\Omega)}\,. \tag{3.15}$$

Then the weighted bound for $\boldsymbol{G}$ in (3.14) follows by substituting (3.7) into each term of this inequality and observing that both terms have the same power of h, whereas the second term has a dominating power of κ. In turn, the weighted bound for Q in (3.14) is obtained by substituting the bound we have just found for $\nabla\boldsymbol{G}$ into (3.1) and observing that the exponent of κ in (3.14) is larger than $\mu/2-1$. ∎

Remark 3.5 Theorem 3.3 and its Corollary 3.4 are also true in a polygon or polyhedron with a milder restriction on the angles than convexity. But we require convexity to guarantee that the components of $\nabla_2\boldsymbol{G}$ belong to $L^2(\Omega)$, and Theorem 3.6 below (that we shall use repeatedly) is meaningless if these components do not belong to $L^2(\Omega)$. ∎

Now we are in a position to establish (0.22).

Theorem 3.6 *Under the assumptions of Theorem 3.3, the weighted estimates (0.22) hold:*

$$\|\sigma^{\mu/2}\nabla_2\boldsymbol{G}\|_{L^2(\Omega)} + \|\sigma^{\mu/2}\nabla\,Q\|_{L^2(\Omega)} \le C\,\kappa^{\mu/2}h^{\lambda/2-1}\,.$$

Proof. Expanding $\nabla_2(\sigma^{\mu/2}\boldsymbol{G})$ and using (2.2), we can write:

$$\|\sigma^{\mu/2}\nabla_2\boldsymbol{G}\|_{L^2(\Omega)} \le \|\nabla_2(\sigma^{\mu/2}\boldsymbol{G})\|_{L^2(\Omega)} + C_1\|\sigma^{\mu/2-1}\nabla\,\boldsymbol{G}\|_{L^2(\Omega)} + C_2\|\sigma^{\mu/2-2}\boldsymbol{G}\|_{L^2(\Omega)}\,. \tag{3.16}$$

Similarly

$$\|\sigma^{\mu/2}\nabla\,Q\|_{L^2(\Omega)} \le \|\nabla(\sigma^{\mu/2}Q)\|_{L^2(\Omega)} + \frac{\mu}{2}\|\sigma^{\mu/2-1}Q\|_{L^2(\Omega)}\,. \tag{3.17}$$

Thus, it remains to find estimates for $\nabla_2(\sigma^{\mu/2}\boldsymbol{G})$ and $\nabla(\sigma^{\mu/2}Q)$. To this end, let us compute the effect of the Stokes operator on $(\sigma^{\mu/2}\boldsymbol{G},\sigma^{\mu/2}Q)$. From (0.12) and (0.13), we infer:

$$\begin{aligned} -\Delta(\sigma^{\mu/2}\boldsymbol{G}) + \nabla(\sigma^{\mu/2}Q) &= -\sigma^{\mu/2}\frac{\partial}{\partial x_j}(\delta_M\boldsymbol{e}_i) - 2(\nabla(\sigma^{\mu/2})\cdot\nabla)\boldsymbol{G} - \Delta(\sigma^{\mu/2})\boldsymbol{G} + \nabla(\sigma^{\mu/2})Q \in L^2(\Omega)^d \\ \operatorname{div}(\sigma^{\mu/2}\boldsymbol{G}) &= \nabla(\sigma^{\mu/2})\cdot\boldsymbol{G} \in H^1_0(\Omega)\,. \end{aligned} \tag{3.18}$$

As $\sigma^{\mu/2}\boldsymbol{G}$ vanishes on $\partial\Omega$, this last equation implies that necessarily $\nabla(\sigma^{\mu/2})\cdot\boldsymbol{G}$ belongs to $H^1_0(\Omega)\cap L^2_0(\Omega)$. Therefore, according to Theorem 0.4 and (0.32), there exists $\boldsymbol{v}\in H^2_0(\Omega)^d$ satisfying (0.31):

$$\operatorname{div}\boldsymbol{v} = \nabla(\sigma^{\mu/2})\cdot\boldsymbol{G}\,,\qquad \|\boldsymbol{v}\|_{H^2(\Omega)} \le C_3|\nabla(\sigma^{\mu/2})\cdot\boldsymbol{G}|_{H^1(\Omega)}\,.$$

Subtracting $\boldsymbol{v}$ from $\sigma^{\mu/2}\boldsymbol{G}$ in (3.18), and thereby utilizing $\operatorname{div}(\sigma^{\mu/2}\boldsymbol{G}-\boldsymbol{v}) = 0$, we infer now from the convexity of Ω that $\sigma^{\mu/2}\boldsymbol{G}\in H^2(\Omega)^d$, $\sigma^{\mu/2}Q\in H^1(\Omega)$ with

$$\begin{aligned} \|\nabla_2(\sigma^{\mu/2}\boldsymbol{G})\|_{L^2(\Omega)} + \|\nabla(\sigma^{\mu/2}Q)\|_{L^2(\Omega)} \le C_4(&\|\sigma^{\mu/2}\nabla\,\delta_M\|_{L^2(\Omega)} \\ &+ \|\sigma^{\mu/2-1}\nabla\,\boldsymbol{G}\|_{L^2(\Omega)} + \|\sigma^{\mu/2-2}\boldsymbol{G}\|_{L^2(\Omega)} + \|\sigma^{\mu/2-1}Q\|_{L^2(\Omega)})\,. \end{aligned}$$

Substituting this inequality into (3.16) and (3.17), we obtain

$$\|\sigma^{\mu/2}\nabla_2\boldsymbol{G}\|_{L^2(\Omega)} + \|\sigma^{\mu/2}\nabla\, Q\|_{L^2(\Omega)} \le C_5(\|\sigma^{\mu/2}\nabla\,\delta_M\|_{L^2(\Omega)} + \|\sigma^{\mu/2-1}\nabla\,\boldsymbol{G}\|_{L^2(\Omega)} + \|\sigma^{\mu/2-2}\boldsymbol{G}\|_{L^2(\Omega)} + \|\sigma^{\mu/2-1}Q\|_{L^2(\Omega)})\,.$$

Then (0.22) follows from this inequality, (2.6), (3.14), (3.7) and the fact that the largest exponent of κ in these estimates is $\mu/2$. ∎

3.2 Weighted interpolation results

Theorem 3.6 enables us to evaluate the approximation error of the spaces X_h and M_h in weighted norms. We shall describe with more precision the approximation operators P_h and r_h, but for the moment, let us assume that $P_h \in \mathcal{L}(H_0^1(\Omega)^d; X_h)$ and $r_h \in \mathcal{L}(L^2(\Omega); \bar{M}_h)$ satisfy the following properties, where the functions of $\bar{M}_h$ are those of M_h without the zero mean-value constraint:

1. P_h and r_h have at least order one and are quasi-local: for all $T \in \mathcal{T}_h$,

$$\|P_h(\boldsymbol{v}) - \boldsymbol{v}\|_{L^2(T)} + h_T\|\nabla(P_h(\boldsymbol{v}) - \boldsymbol{v})\|_{L^2(T)} \le C\,h_T^2\|\nabla_2\boldsymbol{v}\|_{L^2(\Delta_T)}\,, \tag{3.19}$$

$$\|r_h(q) - q\|_{L^2(T)} \le C\,h_T\|\nabla\, q\|_{L^2(\Delta_T)}\,, \tag{3.20}$$

   where Δ_T is a macro-element containing at most L elements of $\mathcal{T}_h$, including T, L being a fixed integer independent of h, q and $\boldsymbol{v}$;

2. P_h preserves the discrete divergence:

$$\int_\Omega q_h \mathrm{div}(P_h(\boldsymbol{v}) - \boldsymbol{v})d\boldsymbol{x} = 0 \quad \forall q_h \in \bar{M}_h\,; \tag{3.21}$$

3. P_h is stable in $H^1(\Omega)$: for all $T \in \mathcal{T}_h$,

$$\|\nabla\, P_h(\boldsymbol{v})\|_{L^2(T)} \le C\|\nabla\,\boldsymbol{v}\|_{L^2(\Delta_T)}\,. \tag{3.22}$$

In the examples we shall use, these properties hold provided $\mathcal{T}_h$ satisfies (1.3).

Remark 3.7 By Fortin's Lemma (cf. Fortin [17] or Girault & Raviart [20]), (3.21) and the global version of (3.22) are equivalent to the uniform inf-sup condition. The additional property of quasi-locality is fundamental here for deriving weighted estimates. ∎

Remark 3.8 Strictly speaking, we should distinguish between the macro-element related to P_h and that related to r_h, especially since r_h is often completely local, in which case its macro-element reduces to T. However, we use the same notation for the sake of simplicity. ∎

Remark 3.9 Note that the mean-value of $r_h(q)$ is not necessarily zero, whatever the mean-value of q. Nevertheless, we shall see that this is not important because (3.21) is equivalent to

$$\int_\Omega q_h \mathrm{div}(P_h(\boldsymbol{v}) - \boldsymbol{v})d\boldsymbol{x} = 0 \quad \forall q_h \in M_h\,.$$

Indeed, $(P_h(\boldsymbol{v}) - \boldsymbol{v}) \in H_0^1(\Omega)^d$ and thus, in this equation, any constant can be added to q_h. ∎

Lemma 3.10 *Suppose P_h and r_h satisfy (3.19)–(3.22). Let $\boldsymbol{v} \in [H^2(\Omega) \cap H^1_0(\Omega)]^d$ and $q \in H^1(\Omega) \cap L^2_0(\Omega)$. For any exponent α, we have:*

$$\|\sigma^{\alpha/2}\nabla(P_h(\boldsymbol{v})-\boldsymbol{v})\|_{L^2(\Omega)} + \kappa\,\|\sigma^{\alpha/2-1}(P_h(\boldsymbol{v})-\boldsymbol{v})\|_{L^2(\Omega)} \le C_1\big(L^{1/2}(2\,L^2+1)^{|\alpha|/4}\big)h\|\sigma^{\alpha/2}\nabla_2\boldsymbol{v}\|_{L^2(\Omega)}\,, \quad (3.23)$$

$$\|\sigma^{\alpha/2}(P_h(\boldsymbol{v})-\boldsymbol{v})\|_{L^2(\Omega)} \le C_2\big(L^{1/2}(2\,L^2+1)^{|\alpha|/4}\big)h^2\|\sigma^{\alpha/2}\nabla_2\boldsymbol{v}\|_{L^2(\Omega)}\,, \quad (3.24)$$

$$\|\sigma^{\alpha/2}(r_h(q)-q)\|_{L^2(\Omega)} \le C_3\big(L^{1/2}(2\,L^2+1)^{|\alpha|/4}\big)h\|\sigma^{\alpha/2}\nabla\, q\|_{L^2(\Omega)}\,. \quad (3.25)$$

Similarly, for $\boldsymbol{v} \in H^1_0(\Omega)^d$ and for any exponent α, we have:

$$\|\sigma^{\alpha/2}\nabla\, P_h(\boldsymbol{v})\|_{L^2(\Omega)} \le C_4\big(L^{1/2}(2\,L^2+1)^{|\alpha|/4}\big)\|\sigma^{\alpha/2}\nabla\,\boldsymbol{v}\|_{L^2(\Omega)}\,. \quad (3.26)$$

Proof. We have:

$$\begin{aligned}
\|\sigma^{\alpha/2}\nabla(P_h(\boldsymbol{v})-\boldsymbol{v})\|^2_{L^2(\Omega)} &= \sum_{T\in\mathcal{T}_h}\int_T \sigma^\alpha|\nabla(P_h(\boldsymbol{v})-\boldsymbol{v})|^2 d\boldsymbol{x} \le C_1\sum_{T\in\mathcal{T}_h} h_T^2 \sup_{\boldsymbol{x}\in T}\sigma^\alpha(\boldsymbol{x})\|\nabla_2\boldsymbol{v}\|^2_{L^2(\Delta_T)}\\
&\le C_1\sum_{T\in\mathcal{T}_h} h_T^2\frac{\sup_{\boldsymbol{x}\in T}\sigma^\alpha(\boldsymbol{x})}{\inf_{\boldsymbol{x}\in\Delta_T}\sigma^\alpha(\boldsymbol{x})}\int_{\Delta_T}\sigma^\alpha|\nabla_2\boldsymbol{v}|^2 d\boldsymbol{x}\\
&\le C_1(2\,L^2+1)^{|\alpha|/2}\sum_{T\in\mathcal{T}_h} h_T^2\int_{\Delta_T}\sigma^\alpha|\nabla_2\boldsymbol{v}|^2 d\boldsymbol{x}\\
&\le C_1 L(2\,L^2+1)^{|\alpha|/2}h^2\|\sigma^{\alpha/2}\nabla_2\boldsymbol{v}\|^2_{L^2(\Omega)}\,,
\end{aligned}$$

applying successively (3.19) and Lemma 2.1 with exponent $|\alpha|$. This is the first part of (3.23). The same proof gives (3.26) and (3.24). Likewise, applying (3.20), we find (3.25). Similarly, by (3.19),

$$\begin{aligned}
\|\sigma^{\alpha/2-1}(P_h(\boldsymbol{v})-\boldsymbol{v})\|^2_{L^2(\Omega)} &\le C_2\sum_{T\in\mathcal{T}_h} h_T^4\sup_{\boldsymbol{x}\in T}\sigma^{\alpha-2}(\boldsymbol{x})\|\nabla_2\boldsymbol{v}\|^2_{L^2(\Delta_T)}\\
&\le C_2\sum_{T\in\mathcal{T}_h} h_T^2\sup_{\boldsymbol{x}\in T}\sigma^\alpha(\boldsymbol{x})\frac{h_T^2}{\inf_{\boldsymbol{x}\in T}\sigma^2(\boldsymbol{x})}\|\nabla_2\boldsymbol{v}\|^2_{L^2(\Delta_T)}\,.
\end{aligned}$$

Then the second part of (3.23) follows from

$$\inf_{\boldsymbol{x}\in\Omega}\sigma(\boldsymbol{x}) \ge \theta = \kappa\, h\,. \quad (3.27)$$

This concludes the proof. ∎

The weighted error estimates for $P_h(\boldsymbol{G})$ and $r_h(Q)$ follow directly from Lemma 3.10 and Theorem 3.6.

Theorem 3.11 *We retain the assumptions of Theorem 3.3 and we assume that P_h and r_h satisfy (3.19)–(3.22). Then*

$$\|\sigma^{\mu/2}\nabla(P_h(\boldsymbol{G})-\boldsymbol{G})\|_{L^2(\Omega)} + \|\sigma^{\mu/2}(r_h(Q)-Q)\|_{L^2(\Omega)} \le C\,\kappa^{\mu/2}h^{\lambda/2}\,, \quad (3.28)$$

$$\|\sigma^{\mu/2}(P_h(\boldsymbol{G})-\boldsymbol{G})\|_{L^2(\Omega)} + h\,\kappa\|\sigma^{\mu/2-1}(P_h(\boldsymbol{G})-\boldsymbol{G})\|_{L^2(\Omega)} \le C\,\kappa^{\mu/2}h^{\lambda/2+1}\,. \quad (3.29)$$

4 Discrete weighted inf-sup condition

We start with a discrete weighted inf-sup condition, which has some intrinsic interest.

Proposition 4.1 *For any $0 < \alpha < d$, there exists a constant $\beta_\alpha > 0$ such that*

$$\beta_\alpha \|\sigma^{\alpha/2} q_h\|_{L^2(\Omega)} \le \sup_{\boldsymbol{v}_h \in X_h} \frac{\int_\Omega q_h \operatorname{div} \boldsymbol{v}_h d\boldsymbol{x}}{\|\sigma^{-\alpha/2} \nabla\, \boldsymbol{v}_h\|_{L^2(\Omega)}} \quad \forall q_h \in M^h. \tag{4.1}$$

Proof. We apply Corollary 2.5 to $q := \sigma^\alpha q_h$ with exponent $-\alpha$: there exists $\boldsymbol{v} \in H^1_0(\Omega)^d$ such that

$$\operatorname{div} \boldsymbol{v} = q - \frac{1}{|\Omega|} \int_\Omega q(\boldsymbol{x}) d\boldsymbol{x} \text{ in } \Omega\,,$$

$$\|\sigma^{-\alpha/2} \nabla\, \boldsymbol{v}\|_{L^2(\Omega)} \le C_{-\alpha} \|\sigma^{-\alpha/2} q\|_{L^2(\Omega)}\,. \tag{4.2}$$

Since q_h has vanishing mean-value, we obtain

$$\|\sigma^{\alpha/2} q_h\|^2_{L^2(\Omega)} = \|\sigma^{-\alpha/2} q\|^2_{L^2(\Omega)} = \int_\Omega q_h q\, d\boldsymbol{x} = \int_\Omega q_h \operatorname{div} \boldsymbol{v}\, d\boldsymbol{x}.$$

In view of (3.21) and (3.26), we can thus write

$$\|\sigma^{\alpha/2} q_h\|_{L^2(\Omega)} = \frac{\int_\Omega q_h \operatorname{div} P_h(\boldsymbol{v})\, d\boldsymbol{x}}{\|\sigma^{\alpha/2} q_h\|_{L^2(\Omega)}} \le C_{-\alpha} \frac{\int_\Omega q_h \operatorname{div} P_h(\boldsymbol{v})\, d\boldsymbol{x}}{\|\sigma^{-\alpha/2} \nabla \boldsymbol{v}\|_{L^2(\Omega)}} \le C'_{-\alpha} \frac{\int_\Omega q_h \operatorname{div} P_h(\boldsymbol{v})\, d\boldsymbol{x}}{\|\sigma^{-\alpha/2} \nabla P_h(\boldsymbol{v})\|_{L^2(\Omega)}}.$$

This proves the assertion. ∎

Considering now (0.25) and (0.26), we propose to establish an estimate for $\sigma^{\alpha/2}(Q_h - r_h(Q))$ for $0 < \alpha < d$, in terms of $\sigma^{\mu/2} \nabla(\boldsymbol{G} - \boldsymbol{G}_h)$. Recall that $\mu = d + \lambda$.

Theorem 4.2 *We retain the assumptions of Theorem 3.3 and we suppose that r_h satisfies (3.20) and P_h satisfies (3.21) and (3.22). For $0 < \alpha < d$, there exists a constant C_α, depending only on α, such that:*

$$\|\sigma^{\alpha/2}(Q_h - r_h(Q))\|_{L^2(\Omega)} \le \frac{C_\alpha}{\theta^{(\mu-\alpha)/2}} \big(\|\sigma^{\mu/2} \nabla(\boldsymbol{G} - \boldsymbol{G}_h)\|_{L^2(\Omega)} + C \kappa^{\mu/2} h^{\lambda/2} \big)\,. \tag{4.3}$$

Proof. Exceptionally, here we need an approximation of Q with zero mean-value. Therefore, using (2.10), we set

$$\rho_h(Q) = r_h(Q) - m(r_h(Q)) = r_h(Q) - m(r_h(Q) - Q)\,.$$

Then

$$\sigma^{\alpha/2}(Q_h - r_h(Q)) = \sigma^{\alpha/2}(Q_h - \rho_h(Q)) - \sigma^{\alpha/2} m(r_h(Q) - Q)\,.$$

By (2.11), (3.27) and (3.28), we have

$$\begin{aligned} \|\sigma^{\alpha/2} m(r_h(Q) - Q)\|_{L^2(\Omega)} &\le C_1 \|\sigma^{\alpha/2}(r_h(Q) - Q)\|_{L^2(\Omega)} = C_1 \Big(\int_\Omega \sigma^{\mu - (\mu - \alpha)} (r_h(Q) - Q)^2 d\boldsymbol{x} \Big)^{1/2} \\ &\le \frac{C_1}{\theta^{(\mu-\alpha)/2}} \|\sigma^{\mu/2}(r_h(Q) - Q)\|_{L^2(\Omega)} \le \frac{C_2}{\theta^{(\mu-\alpha)/2}} \kappa^{\mu/2} h^{\lambda/2}\,. \end{aligned} \tag{4.4}$$

Hence, it remains to deal with $\sigma^{\alpha/2}(Q_h - \rho_h(Q))$. Since $q_h := Q_h - \rho_h(Q) \in M_h$ has zero mean-value, we apply Proposition 4.1 to deduce

$$\beta_\alpha \|\sigma^{\alpha/2}(Q_h - \rho_h(Q))\|_{L^2(\Omega)} \le \sup_{\boldsymbol{v}_h \in X_h} \frac{\int_\Omega (Q_h - \rho_h(Q)) \operatorname{div} \boldsymbol{v}_h d\boldsymbol{x}}{\|\sigma^{-\alpha/2} \nabla \boldsymbol{v}_h\|_{L^2(\Omega)}} = \sup_{\boldsymbol{v}_h \in X_h} \frac{\int_\Omega (Q_h - r_h(Q)) \operatorname{div} \boldsymbol{v}_h d\boldsymbol{x}}{\|\sigma^{-\alpha/2} \nabla \boldsymbol{v}_h\|_{L^2(\Omega)}}. \tag{4.5}$$

Adding and subtracting Q in the numerator for any $\boldsymbol{v}_h$ in X_h, and using the error equation (0.14) for $(\boldsymbol{G}, Q)$, we end up with

$$\int_\Omega (Q_h - r_h(Q)) \operatorname{div} \boldsymbol{v}_h d\boldsymbol{x} = \int_\Omega \nabla(\boldsymbol{G}_h - \boldsymbol{G}) : \nabla \boldsymbol{v}_h d\boldsymbol{x} + \int_\Omega (Q - r_h(Q)) \operatorname{div} \boldsymbol{v}_h d\boldsymbol{x} \,. \tag{4.6}$$

We can bound the first term in (4.6) as follows:

$$\begin{aligned} |\int_\Omega \nabla(\boldsymbol{G}_h - \boldsymbol{G}) : \nabla \boldsymbol{v}_h \, d\boldsymbol{x}| &\le \|\sigma^{\alpha/2}\nabla(\boldsymbol{G}_h - \boldsymbol{G})\|_{L^2(\Omega)} \|\sigma^{-\alpha/2}\nabla \boldsymbol{v}_h\|_{L^2(\Omega)} \\ &\le \frac{1}{\theta^{(\mu-\alpha)/2}} \|\sigma^{\mu/2}\nabla(\boldsymbol{G}_h - \boldsymbol{G})\|_{L^2(\Omega)} \|\sigma^{-\alpha/2}\nabla \, \boldsymbol{v}_h\|_{L^2(\Omega)} \,. \end{aligned} \tag{4.7}$$

Similarly, the second term in (4.6) is bounded by:

$$\begin{aligned} |\int_\Omega (Q - r_h(Q)) \operatorname{div} \boldsymbol{v}_h d\boldsymbol{x}| &\le \frac{\sqrt{d}}{\theta^{(\mu-\alpha)/2}} \|\sigma^{\mu/2}(Q - r_h(Q))\|_{L^2(\Omega)} \|\sigma^{-\alpha/2}\nabla \, \boldsymbol{v}_h\|_{L^2(\Omega)} \\ &\le \frac{C_3}{\theta^{(\mu-\alpha)/2}} \kappa^{\mu/2} h^{\lambda/2} \|\sigma^{-\alpha/2}\nabla \, \boldsymbol{v}_h\|_{L^2(\Omega)} \,. \end{aligned} \tag{4.8}$$

Substituting (4.7) and (4.8) into (4.6), we obtain for any $\boldsymbol{v}_h$ in X_h:

$$\frac{|\int_\Omega (Q_h - r_h(Q)) \operatorname{div} \boldsymbol{v}_h \, d\boldsymbol{x}|}{\|\sigma^{-\alpha/2}\nabla \, \boldsymbol{v}_h\|_{L^2(\Omega)}} \le \frac{1}{\theta^{(\mu-\alpha)/2}} (\|\sigma^{\mu/2}\nabla(\boldsymbol{G} - \boldsymbol{G}_h)\|_{L^2(\Omega)} + C_3 \kappa^{\mu/2} h^{\lambda/2}) \,,$$

and in view of (4.4), (4.3) follows by substituting this inequality into (4.5). ∎

5 General duality argument

This section is devoted to proving (0.29). It estimates $\sigma^{\mu/2+\varepsilon/2-1}(\boldsymbol{G} - \boldsymbol{G}_h)$ in terms of $\sigma^{\mu/2}\nabla(\boldsymbol{G} - \boldsymbol{G}_h)$ for $0 \le \varepsilon \le \varepsilon_0$, where ε_0 is a small positive number that depends on the inner angles of $\partial\Omega$. This estimate is based on the following duality argument, similar to that used by Theorem 3.3, but in contrast to that theorem, it restricts more severely the angles of $\partial\Omega$ in three dimensions. As pointed out in the introduction, the angles should be such that there exists a real number $r > d$ such that whenever $\boldsymbol{f}$ belongs to $L^r(\Omega)^d$ then the solution $(\boldsymbol{v}, q)$ of the Stokes problem

$$-\Delta \, \boldsymbol{v} + \nabla \, q = \boldsymbol{f} \ , \ \operatorname{div} \boldsymbol{v} = 0 \ \text{ in } \Omega \ , \ \boldsymbol{v}|_{\partial\Omega} = \boldsymbol{0} \,,$$

satisfies

$$\boldsymbol{v} \in W^{2,r}(\Omega)^d \ , \ q \in W^{1,r}(\Omega) \,, \tag{5.1}$$

with continuous dependence on $\boldsymbol{f}$.

Theorem 5.1 *Assume that $\mathcal{T}_h$ satisfies (2.5), $\partial\Omega$ is such that (5.1) holds for some real number $r > d$ and P_h and r_h satisfy (3.19)–(3.21). If the numbers $\varepsilon \ge 0$ and $\lambda > 0$ satisfy:*

$$\frac{\lambda}{2} + \varepsilon < 1 - \frac{d}{r} \,, \tag{5.2}$$

then there exists a constant C_ε such that the following bound holds

$$\begin{aligned} \|\sigma^{\mu/2+\varepsilon/2-1}(\boldsymbol{G} - \boldsymbol{G}_h)\|^2_{L^2(\Omega)} \le C_\varepsilon \frac{\theta^\varepsilon}{\kappa} &(\|\sigma^{\mu/2}\nabla(\boldsymbol{G} - \boldsymbol{G}_h)\|_{L^2(\Omega)} + C\kappa^{\mu/2} h^{\lambda/2}) \\ &\times (\|\sigma^{\mu/2}\nabla(\boldsymbol{G} - \boldsymbol{G}_h)\|^2_{L^2(\Omega)} + \|\sigma^{\mu/2-1}(\boldsymbol{G} - \boldsymbol{G}_h)\|^2_{L^2(\Omega)})^{1/2} \,. \end{aligned} \tag{5.3}$$

Proof. Let $(\boldsymbol{\varphi}, s) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$ be the solution of the Stokes problem:

$$-\Delta\,\boldsymbol{\varphi} + \nabla\, s = \sigma^{\mu+\varepsilon-2}(\boldsymbol{G} - \boldsymbol{G}_h)\ ,\ \operatorname{div} \boldsymbol{\varphi} = 0 \text{ in } \Omega\ ,\ \boldsymbol{\varphi} = \boldsymbol{0} \text{ on } \partial\Omega\,. \tag{5.4}$$

By assumption, there exists $r > d$ such that $\boldsymbol{\varphi} \in W^{2,r}(\Omega)^d$, $s \in W^{1,r}(\Omega)$ with

$$\|\boldsymbol{\varphi}\|_{W^{2,r}(\Omega)} + |s|_{W^{1,r}(\Omega)} \le C_r \|\sigma^{\mu+\varepsilon-2}(\boldsymbol{G} - \boldsymbol{G}_h)\|_{L^r(\Omega)}\,. \tag{5.5}$$

Now, multiplying the first equation in (5.4) by $\boldsymbol{G} - \boldsymbol{G}_h$, applying the second equation, the error equation (0.14), (0.13), (0.15) and (3.21), we obtain

$$\begin{aligned}\int_\Omega \sigma^{\mu+\varepsilon-2}|\boldsymbol{G} - \boldsymbol{G}_h|^2 d\boldsymbol{x} &= \int_\Omega (-\Delta\,\boldsymbol{\varphi} + \nabla\, s)\cdot(\boldsymbol{G} - \boldsymbol{G}_h) d\boldsymbol{x} \\ &= \int_\Omega \nabla(\boldsymbol{\varphi} - P_h(\boldsymbol{\varphi})) : \nabla(\boldsymbol{G} - \boldsymbol{G}_h) d\boldsymbol{x} \\ &+ \int_\Omega (Q - r_h(Q))\operatorname{div}(P_h(\boldsymbol{\varphi}) - \boldsymbol{\varphi}) d\boldsymbol{x} \\ &- \int_\Omega (s - r_h(s))\operatorname{div}(\boldsymbol{G} - \boldsymbol{G}_h) d\boldsymbol{x}\,.\end{aligned}$$

Therefore

$$\begin{aligned}\|\sigma^{\mu/2+\varepsilon/2-1}(\boldsymbol{G} - \boldsymbol{G}_h)\|^2_{L^2(\Omega)} \le{}& \sqrt{d}\|\sigma^{-\mu/2}(s - r_h(s))\|_{L^2(\Omega)}\|\sigma^{\mu/2}\nabla(\boldsymbol{G} - \boldsymbol{G}_h)\|_{L^2(\Omega)} \\ &+ \|\sigma^{-\mu/2}\nabla(\boldsymbol{\varphi} - P_h(\boldsymbol{\varphi}))\|_{L^2(\Omega)}(\|\sigma^{\mu/2}\nabla(\boldsymbol{G} - \boldsymbol{G}_h)\|_{L^2(\Omega)} \\ &+ \sqrt{d}\|\sigma^{\mu/2}(Q - r_h(Q))\|_{L^2(\Omega)})\,.\end{aligned} \tag{5.6}$$

Hence applying Lemma 3.10 to the terms involving $\boldsymbol{\varphi}$, s and Q, and using Theorem 3.6, we find that

$$\begin{aligned}\|\sigma^{\mu/2+\varepsilon/2-1}(\boldsymbol{G} - \boldsymbol{G}_h)\|^2_{L^2(\Omega)} \le{}& h\,\|\sigma^{\mu/2}\nabla(\boldsymbol{G} - \boldsymbol{G}_h)\|_{L^2(\Omega)}(C_1\|\sigma^{-\mu/2}\nabla_2\boldsymbol{\varphi}\|_{L^2(\Omega)} \\ &+ C_2\|\sigma^{-\mu/2}\nabla\, s\|_{L^2(\Omega)}) + C_3 h^{1+\lambda/2}\kappa^{\mu/2}\|\sigma^{-\mu/2}\nabla_2\boldsymbol{\varphi}\|_{L^2(\Omega)}\,.\end{aligned} \tag{5.7}$$

Thus it suffices to derive a sharp bound for $\sigma^{-\mu/2}\nabla_2\boldsymbol{\varphi}$ and $\sigma^{-\mu/2}\nabla\, s$. Let us concentrate on $\boldsymbol{\varphi}$, the proof for s being the same. We write

$$\|\sigma^{-\mu/2}\nabla_2\boldsymbol{\varphi}\|^2_{L^2(\Omega)} \le \left(\int_\Omega \sigma^{-\mu\, p'} d\boldsymbol{x}\right)^{1/p'} \left(\int_\Omega |\nabla_2\boldsymbol{\varphi}|^{2p} d\boldsymbol{x}\right)^{1/p}\,, \tag{5.8}$$

with

$$1/p + 1/p' = 1\ ,\ p = r/2 \text{ i.e. } p' = \frac{r}{r-2}\,,$$

where r is the exponent of (5.1). On the one hand, as $p' > 1$, we have $\mu\, p' > d$ and Lemma 1.3 gives:

$$(\int_\Omega \sigma^{-\mu\, p'} d\boldsymbol{x})^{1/p'} \le C_4 \frac{1}{\theta^{\lambda+(2d)/r}}\,.$$

On the other hand, (5.5) yields

$$\left(\int_\Omega |\nabla_2\boldsymbol{\varphi}|^{2p} d\boldsymbol{x}\right)^{1/p} = \|\nabla_2\boldsymbol{\varphi}\|^2_{L^r(\Omega)} \le C_r^2 \|\sigma^{\mu+\varepsilon-2}(\boldsymbol{G} - \boldsymbol{G}_h)\|^2_{L^r(\Omega)}\,.$$

Therefore (5.8) becomes

$$\|\sigma^{-\mu/2}\nabla_2\boldsymbol{\varphi}\|^2_{L^2(\Omega)} \le C_5\frac{1}{\theta^{\lambda+(2d)/r}}\|\sigma^{\mu+\varepsilon-2}(\boldsymbol{G}-\boldsymbol{G}_h)\|^2_{L^r(\Omega)}\,. \tag{5.9}$$

Now, let t be the exponent of the Sobolev imbedding:

$$W^{1,t}(\Omega)\subset L^r(\Omega)\ ,\ \text{ i.e. } t=\frac{r\,d}{r+d}\,,\ 1<t<2 \text{ for } d<r<\frac{2\,d}{d-2}\,,$$

that we can always suppose since r is a little larger than d and we are free to choose r as close to d as we wish. Then, observing that

$$\sigma^{\mu+\varepsilon-2}(\boldsymbol{G}-\boldsymbol{G}_h)\in (H^1_0(\Omega)\cap W^{1,t}(\Omega))^d\,,$$

since $\boldsymbol{G}-\boldsymbol{G}_h$ vanishes on $\partial\Omega$, and setting for simplification

$$\boldsymbol{f}=\sigma^{\mu+\varepsilon-2}(\boldsymbol{G}-\boldsymbol{G}_h)\,,$$

Sobolev's imbedding implies that

$$\|\sigma^{-\mu/2}\nabla_2\boldsymbol{\varphi}\|^2_{L^2(\Omega)} \le C_6\frac{1}{\theta^{\lambda+(2d)/r}}\|\nabla\,\boldsymbol{f}\|^2_{L^t(\Omega)}\,. \tag{5.10}$$

Next, setting $\tau=2/t>1$ owing that $t<2$, we write

$$\begin{aligned}\int_\Omega|\nabla\,\boldsymbol{f}|^t d\boldsymbol{x} &= \int_\Omega \sigma^{(\mu+2\,\varepsilon-4)/\tau}\sigma^{-(\mu+2\,\varepsilon-4)/\tau}|\nabla\,\boldsymbol{f}|^{2/\tau}d\boldsymbol{x}\\ &\le \Big(\int_\Omega \sigma^{(\mu+2\,\varepsilon-4)\tau'/\tau}d\boldsymbol{x}\Big)^{1/\tau'}\Big(\int_\Omega \sigma^{-(\mu+2\,\varepsilon-4)}|\nabla\,\boldsymbol{f}|^2 d\boldsymbol{x}\Big)^{1/\tau}\,,\end{aligned}$$

with $1/\tau+1/\tau'=1$, i.e. $\tau'=2/(2-t)$. We want to apply Lemma 1.3 to the first factor. This is possible if

$$(4-\mu-2\,\varepsilon)\frac{\tau'}{\tau}>d\ \text{ i.e. } r>\frac{d}{1-\lambda/2-\varepsilon}\,.$$

Since $r>d$, this holds provided $\varepsilon\ge 0$ and $\lambda>0$ satisfy (5.2). Then Lemma 1.3 yields:

$$\|\nabla\,\boldsymbol{f}\|^2_{L^t(\Omega)}=\Big(\int_\Omega|\nabla\,\boldsymbol{f}|^t d\boldsymbol{x}\Big)^{2/t}\le C_7\frac{1}{\theta^{2-\lambda-2\,\varepsilon-(2d)/r}}\|\sigma^{2-\mu/2-\varepsilon}\nabla\,\boldsymbol{f}\|^2_{L^2(\Omega)}\,.$$

Expanding the definition of $\boldsymbol{f}$ and using (2.1) with exponent $\mu+\varepsilon-2$, we find

$$\begin{aligned}\sigma^{4-\mu-2\,\varepsilon}|\nabla\,\boldsymbol{f}|^2 &= \sigma^{4-\mu-2\,\varepsilon}|\nabla(\sigma^{\mu+\varepsilon-2}(\boldsymbol{G}-\boldsymbol{G}_h))|^2\\ &\le 2\,\sigma^{\mu}|\nabla(\boldsymbol{G}-\boldsymbol{G}_h)|^2+2(\mu+\varepsilon-2)^2\sigma^{\mu-2}|\boldsymbol{G}-\boldsymbol{G}_h|^2\,.\end{aligned}$$

Therefore

$$\|\nabla\,\boldsymbol{f}\|^2_{L^t(\Omega)}\le C_8\frac{1}{\theta^{2-\lambda-2\,\varepsilon-(2d)/r}}\Big(\|\sigma^{\mu/2}\nabla(\boldsymbol{G}-\boldsymbol{G}_h)\|^2_{L^2(\Omega)}+\|\sigma^{\mu/2-1}(\boldsymbol{G}-\boldsymbol{G}_h)\|^2_{L^2(\Omega)}\Big)\,,$$

substituting into (5.10) and considering that the same argument is applicable to s, this yields:

$$\|\sigma^{-\mu/2}\nabla_2\boldsymbol{\varphi}\|^2_{L^2(\Omega)}+\|\sigma^{-\mu/2}\nabla\,s\|^2_{L^2(\Omega)}\le C_9\frac{\theta^{2\,\varepsilon}}{\theta^2}(\|\sigma^{\mu/2}\nabla(\boldsymbol{G}-\boldsymbol{G}_h)\|^2_{L^2(\Omega)}+\|\sigma^{\mu/2-1}(\boldsymbol{G}-\boldsymbol{G}_h)\|^2_{L^2(\Omega)})\,. \tag{5.11}$$

Then (5.3) follows by substituting (5.11) into (5.7) and using (3.27). ∎

Remark 5.2 We have specified that the constant in (5.3) depends on ε because we shall apply it with different values of ε. Of course, this constant depends also on λ, but we shall only use one value of λ. ∎

The first corollary is derived by choosing $\varepsilon = 0$ in Theorem 5.1.

Corollary 5.3 *We retain the assumptions of Theorem 5.1 with $\varepsilon = 0$. Then*

$$\|\sigma^{\mu/2-1}(\boldsymbol{G} - \boldsymbol{G}_h)\|^2_{L^2(\Omega)} \le \frac{1}{\kappa}(1 + 2C_0^2)\|\sigma^{\mu/2}\nabla(\boldsymbol{G} - \boldsymbol{G}_h)\|^2_{L^2(\Omega)} + C\kappa^{\mu-1}h^{\lambda}\,. \tag{5.12}$$

Proof. Applying (5.3) with $\varepsilon = 0$ and Young's inequality, we obtain:

$$\begin{aligned}\|\sigma^{\mu/2-1}(\boldsymbol{G} - \boldsymbol{G}_h)\|^2_{L^2(\Omega)} &\le \frac{C_0}{\kappa}(\|\sigma^{\mu/2}\nabla(\boldsymbol{G} - \boldsymbol{G}_h)\|_{L^2(\Omega)} + C\kappa^{\mu/2}h^{\lambda/2}) \\ &\quad \times (\|\sigma^{\mu/2}\nabla(\boldsymbol{G} - \boldsymbol{G}_h)\|^2_{L^2(\Omega)} + \|\sigma^{\mu/2-1}(\boldsymbol{G} - \boldsymbol{G}_h)\|^2_{L^2(\Omega)})^{1/2} \\ &\le \frac{1}{2\kappa}\Big(\|\sigma^{\mu/2-1}(\boldsymbol{G} - \boldsymbol{G}_h)\|^2_{L^2(\Omega)} + \|\sigma^{\mu/2}\nabla(\boldsymbol{G} - \boldsymbol{G}_h)\|^2_{L^2(\Omega)} \\ &\quad + 2C_0^2(\|\sigma^{\mu/2}\nabla(\boldsymbol{G} - \boldsymbol{G}_h)\|^2_{L^2(\Omega)} + C^2\kappa^{\mu}h^{\lambda})\Big)\,.\end{aligned}$$

Considering that $\kappa > 1$, we infer (5.12). ∎

The second corollary gives the desired estimate for $\varepsilon > 0$. It follows immediately by substituting (5.12) into (5.3) and applying Young's inequality.

Corollary 5.4 *Under the assumptions of Theorem 5.1, we have:*

$$\|\sigma^{\mu/2+\varepsilon/2-1}(\boldsymbol{G} - \boldsymbol{G}_h)\|^2_{L^2(\Omega)} \le C_\varepsilon\frac{\theta^\varepsilon}{\kappa}\Big(\big(\frac{3}{2} + \frac{1}{2\kappa}(1 + 2C_0^2)\big)\|\sigma^{\mu/2}\nabla(\boldsymbol{G} - \boldsymbol{G}_h)\|^2_{L^2(\Omega)} + C\kappa^{\mu}h^{\lambda}\Big)\,, \tag{5.13}$$

where C_ε is the constant of (5.3) for $\varepsilon > 0$ and C_0 the constant for $\varepsilon = 0$.

6 Super-approximation

This section is devoted to the proof of (0.23) for three popular examples of stable finite element spaces. More generally, we shall prove that if $\boldsymbol{v}_h \in X_h$ and $\boldsymbol{\psi} = \sigma^{\mu}\boldsymbol{v}_h$, then

$$\|\sigma^{-\mu/2}\nabla(\boldsymbol{\psi} - \bar{P}_h(\boldsymbol{\psi}))\|_{L^2(\Omega)} \le C\|\sigma^{\mu/2-1}\boldsymbol{v}_h\|_{L^2(\Omega)}\,. \tag{6.1}$$

This property is based on the fact that

$$\boldsymbol{v}_h = \boldsymbol{p}_k + \boldsymbol{b}\,,$$

where $\boldsymbol{p}_k|_T \in I\!P_k^d$ and $\boldsymbol{b}$ is such that $I_h(\boldsymbol{b}) = \boldsymbol{0}$, where I_h is the standard $I\!P_k$ Lagrange interpolant at the nodes of the principal lattice of degree k in each T. For each example, we shall describe briefly the construction of the approximation operator P_h and its simplified version $\bar{P}_h$, that will be applied to $\boldsymbol{v}_h$.

6.1 Taylor-Hood finite elements

The simplest example is the family of Taylor-Hood $I\!P_k$–$I\!P_{k-1}$ finite elements where $\boldsymbol{b} = \boldsymbol{0}$. In two dimensions, for any integer $k \ge 2$, Taylor-Hood finite elements have a quasi-local interpolation operator P_h satisfying (3.19), (3.21) and (3.22) and in three dimensions, this is true for $k \ge 3$ (cf. [22]). In three dimensions, if $k = 2$, this also holds if $\mathcal{T}_h$ consists in hexahedra, each hexahedra being split into twelve tetrahedra (cf. Ciarlet Jr. & Girault [12]). Let us study this case, the others being simpler.

The finite element spaces are:

$$X_h = \{\boldsymbol{v}_h \in \mathcal{C}^0(\overline{\Omega})^3\,;\, \boldsymbol{v}_h|_T \in I\!P_2^3 \quad \forall T \in \mathcal{T}_h\,\} \cap H_0^1(\Omega)^3\,, \tag{6.2}$$

$$\bar{M}_h = \{q_h \in \mathcal{C}^0(\overline{\Omega})\,;\, \boldsymbol{v}_h|_T \in I\!P_1 \quad \forall T \in \mathcal{T}_h\,\}\,, \quad M_h = \bar{M}_h \cap L_0^2(\Omega)\,. \tag{6.3}$$

The construction of P_h proposed in [22] proceeds in two steps: first it constructs an auxiliary operator R_h that preserves the mean value of the divergence, which is a weak form of (3.21), and then it adds a correction to R_h so that P_h satisfies (3.21). For the Taylor-Hood finite elements, the correction is computed locally on macro-elements, with or without overlaps, by a procedure that generalizes that of Boland and Nicolaides [6] and Stenberg [38]. In all cases except $d = 3$ and $k = 2$, the auxiliary operator R_h can be easily constructed quasi-locally and the mean-value of the divergence is preserved in each element. This is made possible because these elements have at least one degree of freedom in the interior of each face. This is not the case when $d = 3$ and $k = 2$, where all degrees of freedom are located on edges. In contrast, the above hexahedral structure has one degree of freedom on each of its faces; for this reason we ask that $\mathcal{T}_h$ have this structure.

Let us describe first $\bar{P}_h$; the operator P_h will be easily deduced from it. Let $\{\mathcal{O}_i\}_{1\le i\le R}$ be the family of hexahedra partitioning $\mathcal{T}_h$. Note that each face F of $\mathcal{O}_i$ is subdivided into two triangles along one of its diagonals, say d_F, each triangle being a face of a tetrahedron contained in $\mathcal{O}_i$.

For the *first step*, in each $\mathcal{O}_i$, we define

$$\bar{R}_h(\boldsymbol{\psi}) = I_h(\boldsymbol{\psi}) + \sum_{F\subset\partial\mathcal{O}_i} \boldsymbol{c}_F b_F\,, \tag{6.4}$$

where I_h is the $I\!P_2$ Lagrange interpolant and b_F is the polynomial of degree two in each T that takes the value 1 at the midpoint of the diagonal d_F and 0 at all the other nodes of the principal lattice of degree 2. This degree of freedom at the midpoint of the diagonal is used to preserve the mean-value of the divergence on $\mathcal{O}_i$. Indeed, we define $\boldsymbol{c}_F$ by:

$$\boldsymbol{c}_F = -\frac{1}{\int_F b_F ds}\int_F (I_h(\boldsymbol{\psi}) - \boldsymbol{\psi})ds\,, \tag{6.5}$$

and thus

$$\int_{\mathcal{O}_i} \mathrm{div}(\bar{R}_h(\boldsymbol{\psi}) - \boldsymbol{\psi})d\boldsymbol{x} = 0\,.$$

For the *second step*, in each $\mathcal{O}_i$, we define the local spaces:

$$X_h(\mathcal{O}_i) = \{\boldsymbol{v}_h \in X_h\,;\, \boldsymbol{v}_h|_{\partial\mathcal{O}_i} = \boldsymbol{0}\}\,,$$

$$M_h(\mathcal{O}_i) = \{q_h|_{\mathcal{O}_i} - \frac{1}{|\mathcal{O}_i|}\int_{\mathcal{O}_i} q_h(\boldsymbol{x})d\boldsymbol{x}\,;\, q_h \in M_h\}\,.$$

Then, following the argument of [22], we can construct $c_h(\boldsymbol{\psi}) \in X_h(\mathcal{O}_i)$ such that

$$\int_{\mathcal{O}_i} q_h \mathrm{div}\, c_h(\boldsymbol{\psi})d\boldsymbol{x} = \int_{\mathcal{O}_i} q_h \mathrm{div}(\boldsymbol{\psi} - \bar{R}_h(\boldsymbol{\psi}))d\boldsymbol{x} \quad \forall q_h \in M_h(\mathcal{O}_i)\,, \tag{6.6}$$

$$\|\nabla\, c_h(\boldsymbol{\psi})\|_{L^2(\mathcal{O}_i)} \le \frac{1}{\eta}\|\mathrm{div}(\boldsymbol{\psi} - \bar{R}_h(\boldsymbol{\psi})\|_{L^2(\mathcal{O}_i)}\,, \tag{6.7}$$

with a constant $\eta > 0$ independent of i, h and $\boldsymbol{\psi}$. Finally, as the macro-elements $\mathcal{O}_i$ form a partition of $\mathcal{T}_h$ we set:

$$\bar{P}_h(\boldsymbol{\psi}) = \bar{R}_h(\boldsymbol{\psi}) + c_h(\boldsymbol{\psi})\,. \tag{6.8}$$

Remark 6.1 The only difference between P_h and $\bar{P}_h$ is that for defining P_h in (6.4), one must replace I_h by a regularization operator such as the one proposed by Scott & Zhang [36]. Here we can use I_h because on one hand $\boldsymbol{\psi}$ is continuous and on the other hand we do not need (3.22) for proving (0.23). ∎

Proposition 6.2 *Assume that $\mathcal{T}_h$ satisfies (1.3). Let $h_i = \sup_{T\subset\mathcal{O}_i} h_T$. We have*

$$\|\nabla(\boldsymbol{\psi} - \bar{R}_h(\boldsymbol{\psi}))\|_{L^2(\mathcal{O}_i)} \le Ch_i^2(\sum_{T\subset\mathcal{O}_i} \|\nabla_3\boldsymbol{\psi}\|^2_{L^2(T)})^{1/2}\,. \tag{6.9}$$

Proof. By definition, we have

$$\|\nabla(\boldsymbol{\psi} - \bar{R}_h(\boldsymbol{\psi}))\|_{L^2(\mathcal{O}_i)} \le \|\nabla(\boldsymbol{\psi} - I_h(\boldsymbol{\psi}))\|_{L^2(\mathcal{O}_i)} + \sum_{F\subset\partial\,\mathcal{O}_i} |\boldsymbol{c}_F|\|\nabla\, b_F\|_{L^2(\mathcal{O}_i)}\,.$$

First, as I_h is the Lagrange interpolation operator, it is local to each T and hence its standard approximation properties and the regularity of $\mathcal{T}_h$ yield:

$$\|\boldsymbol{\psi} - I_h(\boldsymbol{\psi})\|_{L^2(T)} + h_T\|\nabla(\boldsymbol{\psi} - I_h(\boldsymbol{\psi}))\|_{L^2(T)} \le C_1 h_T^3\|\nabla_3\boldsymbol{\psi}\|_{L^2(T)}\,. \tag{6.10}$$

Next, an easy calculation on the reference element gives:

$$\|\nabla\, b_F\|_{L^2(\mathcal{O}_i)} \le \frac{C_2}{\varrho_i}|T_i|^{1/2}\,,$$

where

$$\varrho_i = \inf_{T\subset\mathcal{O}_i} \varrho_T\ ,\ \ |T_i| = \sup_{T\subset\mathcal{O}_i} |T|\,.$$

Finally, by applying the trace theorem on the reference element, we readily infer that:

$$|\boldsymbol{c}_F| \le \frac{C_3}{|F|^{1/2}}\|\boldsymbol{\psi} - I_h(\boldsymbol{\psi})\|_{L^2(F)} \le C_4(|T_1|^{-1/2}h_{T_1}^3\|\nabla_3\boldsymbol{\psi}\|_{L^2(T_1)} + |T_2|^{-1/2}h_{T_2}^3\|\nabla_3\boldsymbol{\psi}\|_{L^2(T_2)})\,,$$

where T_1 and T_2 are the two tetrahedra of $\mathcal{O}_i$ sharing the face F. Collecting these two inequalities and using the fact that (1.3) implies that $\mathcal{T}_h$ is locally uniformly regular (i.e. h_i/ϱ_i is bounded independently of i and h), we obtain

$$|\boldsymbol{c}_F|\|\nabla\, b_F\|_{L^2(\mathcal{O}_i)} \le C_5 h_i^2 \sum_{T\subset\mathcal{O}_i} \|\nabla_3\boldsymbol{\psi}\|^2_{L^2(T)})^{1/2}\,.$$

Hence (6.9) follows from this inequality and (6.10). ∎

Theorem 6.3 *Assume that $\mathcal{T}_h$ satisfies (1.3); then (6.1) holds for $\bar{P}_h$ defined by (6.8):*

$$\|\sigma^{-\mu/2}\nabla(\boldsymbol{\psi} - \bar{P}_h(\boldsymbol{\psi}))\|_{L^2(\Omega)} \le C\|\sigma^{\mu/2-1}\boldsymbol{v}_h\|_{L^2(\Omega)}\,.$$

Proof. From the definition of $\bar{P}_h$, (6.7) and (6.9), we readily deduce that

$$\|\nabla(\boldsymbol{\psi} - \bar{P}_h(\boldsymbol{\psi}))\|_{L^2(\mathcal{O}_i)} \le \|\nabla(\boldsymbol{\psi} - \bar{R}_h(\boldsymbol{\psi}))\|_{L^2(\mathcal{O}_i)}(1 + \frac{\sqrt{d}}{\eta}) \le C_1 h_i^2\left(\sum_{T\subset\mathcal{O}_i} \|\nabla_3\boldsymbol{\psi}\|^2_{L^2(T)}\right)^{1/2}\,.$$

Therefore,

$$\|\sigma^{-\mu/2}\nabla(\boldsymbol{\psi}-\bar{P}_h(\boldsymbol{\psi}))\|^2_{L^2(\Omega)} = \sum_{i=1}^{R}\int_{\mathcal{O}_i}\sigma^{-\mu}|\nabla(\boldsymbol{\psi}-\bar{P}_h(\boldsymbol{\psi}))|^2 d\boldsymbol{x} \le \sum_{i=1}^{R}\sup_{\boldsymbol{x}\in\mathcal{O}_i}\sigma^{-\mu}(\boldsymbol{x})\int_{\mathcal{O}_i}|\nabla(\boldsymbol{\psi}-\bar{P}_h(\boldsymbol{\psi}))|^2 d\boldsymbol{x}$$
$$\le C_1^2\sum_{i=1}^{R}\sup_{\boldsymbol{x}\in\mathcal{O}_i}\sigma^{-\mu}(\boldsymbol{x})\int_{\mathcal{O}_i} h_i^4\sum_{T\subset\mathcal{O}_i}\|\nabla_3\boldsymbol{\psi}\|^2_{L^2(T)}\,.$$

Hence we have to estimate $\nabla_3\boldsymbol{\psi}$. Here the crucial result is that, applying (2.2), we have in each T

$$|\nabla_3\boldsymbol{\psi}| = |\nabla_3(\sigma^\mu\boldsymbol{v}_h)| \le c_1|\sigma^{\mu-1}\nabla_2\boldsymbol{v}_h| + c_2|\sigma^{\mu-2}\nabla\,\boldsymbol{v}_h| + c_3|\sigma^{\mu-3}\boldsymbol{v}_h|\,, \tag{6.11}$$

because each component of $\nabla_3\boldsymbol{v}_h$ is zero since $\boldsymbol{v}_h$ belongs to $I\!P_2^3$. First,

$$\|\sigma^{\mu-3}\boldsymbol{v}_h\|^2_{L^2(T)} = \int_T\sigma^\mu\,\sigma^{\mu-6}|\boldsymbol{v}_h|^2 d\boldsymbol{x} \le \sup_{\boldsymbol{x}\in T}\sigma^\mu(\boldsymbol{x})\|\sigma^{\mu/2-3}\boldsymbol{v}_h\|^2_{L^2(T)}\,.$$

Next, by applying an inverse inequality to $\nabla_2\boldsymbol{v}_h$ in T (that is valid because $I\!P_2^3$ is a finite-dimensional space), we obtain

$$\|\sigma^{\mu-1}\nabla_2\boldsymbol{v}_h\|^2_{L^2(T)} = \int_T\sigma^\mu\,\sigma^{\mu-2}|\nabla_2\boldsymbol{v}_h|^2 d\boldsymbol{x} \le \frac{c_4}{\varrho_T^4}\sup_{\boldsymbol{x}\in T}\sigma^\mu(\boldsymbol{x})\|\sigma^{\mu/2-1}\boldsymbol{v}_h\|^2_{L^2(T)}\,,$$

and similarly,

$$\|\sigma^{\mu-2}\nabla\,\boldsymbol{v}_h\|^2_{L^2(T)} \le \sup_{\boldsymbol{x}\in T}\sigma^\mu(\boldsymbol{x})\int_T\sigma^{\mu-4}|\nabla\,\boldsymbol{v}_h|^2 d\boldsymbol{x} \le \frac{c_5}{\varrho_T^2}\sup_{\boldsymbol{x}\in T}\sigma^\mu(\boldsymbol{x})\|\sigma^{\mu/2-2}\boldsymbol{v}_h\|^2_{L^2(T)}\,.$$

Then (6.1) follows from Lemma 2.1, (1.3) and (3.27). ∎

Remark 6.4 The above argument extends straighforwardly to Taylor-Hood finite elements of higher degree in two and three dimensions. It is simpler because the first step can be performed locally in each T. The macro-elements for the second step are "stars" of elements that share the same vertex, cf. [22]. ∎

Finally, as far as the pressure is concerned, we choose for r_h a regularization operator such as proposed in [36].

6.2 The "mini" element

For the "mini" element, the discrete pressure space is defined by (6.3) and the discrete velocity space is the space of continuous functions $\boldsymbol{v}_h$ defined in each T by (cf. Arnold, Brezzi & Fortin [3] or [20])

$$\boldsymbol{v}_h = \sum_{i=1}^{d+1}\boldsymbol{v}_i\lambda_i + \boldsymbol{v}_c b_T = I_h(\boldsymbol{v}_h) + \boldsymbol{v}_c b_T\,, \tag{6.12}$$

where $\boldsymbol{v}_i$ are the values of $\boldsymbol{v}_h$ at the vertices $\boldsymbol{a}_i$ of T, λ_i are the barycentric coordinates of T, I_h is the $I\!P_1$ Lagrange interpolant at the vertices of T,

$$b_T = \prod_{i=1}^{d+1}\lambda_i\ ,\ \ \boldsymbol{v}_c = \boldsymbol{v}_h(\boldsymbol{c}) - I_h(\boldsymbol{v}_h)(\boldsymbol{c})\,,$$

with $\boldsymbol{c}$ the center of T. Then

$$\bar{P}_h(\boldsymbol{\psi}) = I_h(\boldsymbol{\psi}) + \sum_{T \subset \mathcal{T}_h} \boldsymbol{c}_T b_T \,, \tag{6.13}$$

where

$$\boldsymbol{c}_T = \frac{1}{\int_T b_T \, d\boldsymbol{x}} \int_T (\boldsymbol{\psi} - I_h(\boldsymbol{\psi})) d\boldsymbol{x} \,. \tag{6.14}$$

Note that $I_h(b_T) = 0$; thus setting $\boldsymbol{p}_1 = I_h(\boldsymbol{v}_h) \in I\!P_1^3$, we have:

$$\boldsymbol{\psi} - I_h(\boldsymbol{\psi}) = \sigma^\mu \boldsymbol{p}_1 + \sigma^\mu \boldsymbol{v}_c b_T - I_h(\sigma^\mu \boldsymbol{p}_1) \,. \tag{6.15}$$

Lemma 6.5 *Let $\mathcal{T}_h$ satisfy (1.3); then*

$$\|\sigma^{-\mu/2} \nabla(\sigma^\mu \boldsymbol{p}_1 - I_h(\sigma^\mu \boldsymbol{p}_1))\|_{L^2(T)} \le C \, \|\sigma^{\mu/2-1} \boldsymbol{p}_1\|_{L^2(T)} \,. \tag{6.16}$$

We skip the proof, as well as that of the next lemma, since they are straightforward.

Lemma 6.6 *Let $\mathcal{T}_h$ satisfy (1.3). In each T, for any function f, set*

$$m_T(f) = \frac{1}{\int_T b_T \, d\boldsymbol{x}} \int_T b_T(\boldsymbol{x}) f(\boldsymbol{x}) \, d\boldsymbol{x} \,. \tag{6.17}$$

Then

$$\|\sigma^{-\mu/2} \nabla[(\sigma^\mu - m_T(\sigma^\mu)) b_T]\|_{L^2(T)} \le C \, \|\sigma^{\mu/2-1}\|_{L^2(T)} \,. \tag{6.18}$$

Theorem 6.7 *Let $\mathcal{T}_h$ satisfy (1.3); then (6.1) holds for $\bar{P}_h$ defined by (6.13):*

$$\|\sigma^{-\mu/2} \nabla(\boldsymbol{\psi} - \bar{P}_h(\boldsymbol{\psi}))\|_{L^2(\Omega)} \le C \|\sigma^{\mu/2-1} \boldsymbol{v}_h\|_{L^2(\Omega)} \,.$$

Proof. From the definition (6.13) we derive in each T:

$$\boldsymbol{\psi} - \bar{P}_h(\boldsymbol{\psi}) = \boldsymbol{\psi} - I_h(\boldsymbol{\psi}) - \frac{b_T}{\int_T b_T \, d\boldsymbol{x}} \int_T (\boldsymbol{\psi} - I_h(\boldsymbol{\psi})) d\boldsymbol{x} \,.$$

Thus, (6.15) implies

$$\boldsymbol{\psi} - \bar{P}_h(\boldsymbol{\psi}) = \sigma^\mu \boldsymbol{p}_1 - I_h(\sigma^\mu \boldsymbol{p}_1) + \boldsymbol{v}_c b_T (\sigma^\mu - m_T(\sigma^\mu)) - \frac{b_T}{\int_T b_T \, d\boldsymbol{x}} \int_T (\sigma^\mu \boldsymbol{p}_1 - I_h(\sigma^\mu \boldsymbol{p}_1)) d\boldsymbol{x} \,.$$

Therefore

$$\begin{aligned} \|\sigma^{-\mu/2} \nabla(\boldsymbol{\psi} - \bar{P}_h(\boldsymbol{\psi}))\|_{L^2(T)} \le {} & \|\sigma^{-\mu/2} \nabla(\sigma^\mu \boldsymbol{p}_1 - I_h(\sigma^\mu \boldsymbol{p}_1))\|_{L^2(T)} \\ & + |\boldsymbol{v}_c| \|\sigma^{-\mu/2} \nabla[(\sigma^\mu - m_T(\sigma^\mu)) b_T]\|_{L^2(T)} \\ & + \Big| \frac{1}{\int_T b_T \, d\boldsymbol{x}} \int_T (\sigma^\mu \boldsymbol{p}_1 - I_h(\sigma^\mu \boldsymbol{p}_1)) d\boldsymbol{x} \Big| \|\sigma^{-\mu/2} \nabla \, b_T\|_{L^2(T)} \,. \end{aligned} \tag{6.19}$$

But

$$\|\sigma^{-\mu/2} \nabla \, b_T\|_{L^2(T)}^2 \le C \sup_{\boldsymbol{x} \in T} \sigma^{-\mu}(\boldsymbol{x}) \frac{|T|}{\varrho_T^2} \,. \tag{6.20}$$

Then substituting the bounds (6.16), (6.18) and (6.20) into (6.19), and reverting to $\hat{T}$, we obtain

$$\|\sigma^{-\mu/2} \nabla(\boldsymbol{\psi} - \bar{P}_h(\boldsymbol{\psi}))\|_{L^2(T)}^2 \le c_1 |T| \int_{\hat{T}} \hat{\sigma}^{\mu-2} (|\hat{\boldsymbol{p}}_1|^2 + |\hat{\boldsymbol{v}}_c|^2) d\hat{\boldsymbol{x}} \,. \tag{6.21}$$

But $\boldsymbol{p}_1$ and $\boldsymbol{v}_c$ are invariant under an affine transformation, i.e.

$$\hat{\boldsymbol{p}}_1 = \hat{I}(\hat{\boldsymbol{v}}) \ , \ \hat{\boldsymbol{v}}_c = \hat{\boldsymbol{v}}(\hat{\boldsymbol{c}}) - \hat{I}(\hat{\boldsymbol{c}}) \, ,$$

where $\hat{I}$ is the $\mathbb{P}_1$ interpolation operator at the vertices of $\hat{T}$ and $\hat{\boldsymbol{c}}$ is the center of $\hat{T}$. Hence the mapping:

$$\hat{\boldsymbol{v}} \mapsto \left(\int_{\hat{T}} \hat{\sigma}^{\mu-2} (|\hat{\boldsymbol{p}}_1|^2 + |\hat{\boldsymbol{v}}_c|^2) d\hat{\boldsymbol{x}} \right)^{1/2} ,$$

is a norm for $\hat{\boldsymbol{v}}$ and therefore

$$\int_{\hat{T}} \hat{\sigma}^{\mu-2} (|\hat{\boldsymbol{p}}_1|^2 + |\hat{\boldsymbol{v}}_c|^2) d\hat{\boldsymbol{x}} \leq c_2 \int_{\hat{T}} \hat{\sigma}^{\mu-2} |\hat{\boldsymbol{v}}|^2 d\hat{\boldsymbol{x}} \, .$$

With (6.21), this proves (6.1). ∎

Finally, r_h is defined as in Section 6.1.

Remark 6.8 Similar super-approximation properties for the "mini" element in two dimensions are established by Gastaldi & Nochetto in [18] and by Arnold & Liu in [4]. ∎

6.3 The Bernardi-Raugel element

For the Bernardi-Raugel element, the pressure space is defined by:

$$\bar{M}_h = \{q_h \in L^2(\Omega) \, ; \, q_h|_T \in \mathbb{P}_0 \quad \forall T \in \mathcal{T}_h\} \ , \ M_h = \bar{M}_h \cap L^2_0(\Omega) \, . \tag{6.22}$$

As far as the velocity is concerned, let F denote any one of the $d+1$ faces of an element T, $\boldsymbol{c}_F$ the center of F and $\boldsymbol{n}_F$ the unit normal to F exterior to T. Let b_F denote the polynomial of degree d that vanishes on $\partial T \setminus F$ and takes the value 1 at the center c_F of F (e.g. if F lies on the plane $\lambda_1 = 0$ in three dimensions, then $b_F = 27\lambda_2\lambda_3\lambda_4$). Then $\boldsymbol{v}_h$ is defined in each T by (cf. Bernardi & Raugel [5] or [20]):

$$\boldsymbol{v}_h = \sum_{i=1}^{d+1} \boldsymbol{v}_i \lambda_i + \sum_{F \in \partial T} (\boldsymbol{v}_F \cdot \boldsymbol{n}_F) b_F \boldsymbol{n}_F = I_h(\boldsymbol{v}_h) + \sum_{F \in \partial T} (\boldsymbol{v}_F \cdot \boldsymbol{n}_F) b_F \boldsymbol{n}_F \, , \tag{6.23}$$

where

$$\boldsymbol{v}_F \cdot \boldsymbol{n}_F = \boldsymbol{v}_h(\boldsymbol{c}_F) \cdot \boldsymbol{n}_F - (I_h(\boldsymbol{v}_h)(\boldsymbol{c}_F)) \cdot \boldsymbol{n}_F \, .$$

Note that (6.23) does not depend on the orientation of $\boldsymbol{n}_F$. Then $\bar{P}_h(\boldsymbol{\psi})$ is defined by

$$\bar{P}_h(\boldsymbol{\psi}) = I_h(\boldsymbol{\psi}) + \sum_{F \in \partial T} \left(\frac{1}{\int_F b_F \, ds} \int_F (\boldsymbol{\psi} - I_h(\boldsymbol{\psi})) \cdot \boldsymbol{n}_F ds \right) b_F \boldsymbol{n}_F \, , \tag{6.24}$$

so that for all segments F:

$$\int_F (\boldsymbol{\psi} - \bar{P}_h(\boldsymbol{\psi})) \cdot \boldsymbol{n}_F ds = 0 \, .$$

Note that $I_h(b_F) = 0$ since b_F vanishes at the vertices of T; thus with the above notation for $\boldsymbol{p}_1$, we have:

$$\boldsymbol{\psi} - I_h(\boldsymbol{\psi}) = \sigma^\mu \boldsymbol{p}_1 + \sum_{F \in \partial T} \sigma^\mu (\boldsymbol{v}_F \cdot \boldsymbol{n}_F) b_F \boldsymbol{n}_F - I_h(\sigma^\mu \boldsymbol{p}_1) \, . \tag{6.25}$$

Theorem 6.9 *Let $\mathcal{T}_h$ satisfy (1.3); then (6.1) holds for $\bar{P}_h$ defined by (6.24):*

$$\|\sigma^{-\mu/2}\nabla(\boldsymbol{\psi} - \bar{P}_h(\boldsymbol{\psi}))\|_{L^2(\Omega)} \le C\|\sigma^{\mu/2-1}\boldsymbol{v}_h\|_{L^2(\Omega)}\,.$$

Proof. Here, we set

$$m_F(f) = \frac{1}{\int_F b_F\,ds}\int_F b_F(s)f(s)\,ds\,. \tag{6.26}$$

It stems from (6.25) that

$$\begin{aligned}\boldsymbol{\psi} - \bar{P}_h(\boldsymbol{\psi}) = \sigma^\mu\boldsymbol{p}_1 - I_h(\sigma^\mu\boldsymbol{p}_1) + \sum_{F\in\partial T}(\boldsymbol{v}_F\cdot\boldsymbol{n}_F)b_F\boldsymbol{n}_F(\sigma^\mu - m_F(\sigma^\mu))\\ - \sum_{F\in\partial T}[\frac{1}{\int_F b_F\,ds}\int_F(\sigma^\mu\boldsymbol{p}_1 - I_h(\sigma^\mu\boldsymbol{p}_1))\cdot\boldsymbol{n}_F ds]b_F\boldsymbol{n}_F\,.\end{aligned}$$

The contribution of the first term is estimated in (6.16). For the second term, observe that again

$$m_F(f) = m_{\hat{F}}(\hat{f}) := \frac{1}{\int_{\hat{F}} b_{\hat{F}}\,d\hat{s}}\int_{\hat{F}} b_{\hat{F}}\,\hat{f}\,d\hat{s}\,.$$

Therefore, the argument of Lemma 6.6 gives:

$$\|\sigma^{-\mu/2}\nabla[(\sigma^\mu - m_F(\sigma^\mu))b_F]\|_{L^2(T)} \le C_1\,\|\sigma^{\mu/2-1}\|_{L^2(T)}\,.$$

Then as in the preceding theorem, we derive in each T:

$$\sum_{F\in\partial T}\|\sigma^{-\mu/2}(\boldsymbol{v}_F\cdot\boldsymbol{n}_F)\nabla[b_F\boldsymbol{n}_F(\sigma^\mu - m_F(\sigma^\mu))]\|_{L^2(T)} \le C_2(\sum_{F\in\partial T}|\boldsymbol{v}_F\cdot\boldsymbol{n}_F|)\|\sigma^{\mu/2-1}\|_{L^2(T)}\,. \tag{6.27}$$

As far as the contribution of the third term is concerned, as $\sigma^\mu\boldsymbol{p}_1 - I_h(\sigma^\mu\boldsymbol{p}_1)$ belongs to a finite-dimensional space and I_h preserves $\mathbb{P}_1$, we have on one hand

$$\begin{aligned}[\frac{1}{\int_F b_F\,ds}\int_F(\sigma^\mu\boldsymbol{p}_1 - I_h(\sigma^\mu\boldsymbol{p}_1))\cdot\boldsymbol{n}_F ds| &\le C_3|T|^{-1/2}\|\sigma^\mu\boldsymbol{p}_1 - I_h(\sigma^\mu\boldsymbol{p}_1)\|_{L^2(T)}\\ &\le C_4|T|^{-1/2}h_T^2\|\nabla_2(\sigma^\mu\boldsymbol{p}_1)\|_{L^2(T)}\,.\end{aligned}$$

On the other hand, we have the analogue of (6.20)

$$\|\sigma^{-\mu/2}\nabla\,b_F\|^2_{L^2(T)} \le C_5\sup_{\boldsymbol{x}\in T}\sigma^{-\mu}(\boldsymbol{x})\frac{|T|}{\varrho_T^2}\,.$$

Hence

$$\begin{aligned}\sum_{F\in\partial T}|\frac{1}{\int_F b_F\,ds}\int_F(\sigma^\mu\boldsymbol{p}_1 - I_h(\sigma^\mu\boldsymbol{p}_1))\cdot\boldsymbol{n}_F ds|\|\sigma^{-\mu/2}\nabla\,b_F\|_{L^2(T)}\\ \le C_6\sup_{\boldsymbol{x}\in T}\sigma^{-\mu/2}(\boldsymbol{x})h_T\|\nabla_2(\sigma^\mu\boldsymbol{p}_1)\|_{L^2(T)} \le C_7\|\sigma^{\mu/2-1}\boldsymbol{p}_1\|_{L^2(T)}\,.\end{aligned} \tag{6.28}$$

Therefore, collecting (6.16), (6.27) and (6.28), we obtain

$$\|\sigma^{-\mu/2}\nabla(\boldsymbol{\psi} - \bar{P}_h(\boldsymbol{\psi}))\|^2_{L^2(T)} \le C_8\int_T\sigma^{\mu-2}(|\boldsymbol{p}_1|^2 + \sum_{F\in\partial T}|\boldsymbol{v}_F\cdot\boldsymbol{n}_F|^2)d\boldsymbol{x}\,, \tag{6.29}$$

and it remains to prove that the mapping

$$\boldsymbol{v}_h \mapsto \left(\int_T \sigma^{\mu-2}(|\boldsymbol{p}_1|^2 + \sum_{F\in\partial T} |\boldsymbol{v}_F \cdot \boldsymbol{n}_F|^2) d\boldsymbol{x}\right)^{1/2},$$

is a norm for $\boldsymbol{v}_h$ in T uniformly equivalent to $\|\sigma^{\mu/2-1}\boldsymbol{v}_h\|_{L^2(T)}$.

Passing to the reference element, we have

$$\hat{\boldsymbol{v}} = \hat{I}(\hat{\boldsymbol{v}}) + \sum_{\hat{F}\in\partial\hat{T}} \widehat{\boldsymbol{v}_F \cdot \boldsymbol{n}_F} b_{\hat{F}} \frac{(B^{-1})^T \hat{\boldsymbol{n}}_{\hat{F}}}{|(B^{-1})^T \hat{\boldsymbol{n}}_{\hat{F}}|}.$$

Clearly, since $(B^{-1})^T \hat{\boldsymbol{n}}_{\hat{F}} \neq \boldsymbol{0}$, then $\hat{\boldsymbol{v}} = \boldsymbol{0}$ if and only if $\hat{\boldsymbol{v}}(\hat{\boldsymbol{a}}_i) = \boldsymbol{0}$ for $1 \le i \le d+1$ and $\widehat{\boldsymbol{v}_F \cdot \boldsymbol{n}_F} = 0$ for all faces $\hat{F}$ of $\hat{T}$. Therefore the mapping

$$\hat{\boldsymbol{v}} \mapsto \left(\int_{\hat{T}} \hat{\sigma}^{\mu-2}(\sum_{i=1}^{d+1} |\hat{\boldsymbol{v}}(\hat{\boldsymbol{a}}_i)|^2 + \sum_{\hat{F}\in\partial\hat{T}} |\widehat{\boldsymbol{v}_F \cdot \boldsymbol{n}_F}|^2) d\hat{\boldsymbol{x}}\right)^{1/2},$$

is a norm on the space generated by $I\!P_1^d$ and $b_{\hat{F}}$ for all faces $\hat{F}$ of $\hat{T}$, space on which all the norms are equivalent. As a consequence

$$\begin{aligned}
(\int_T \sigma^{\mu-2}(|\boldsymbol{p}_1|^2 + \sum_{F\in\partial T} |\boldsymbol{v}_F \cdot \boldsymbol{n}_F|^2) d\boldsymbol{x})^{1/2} &= \frac{|T|^{1/2}}{|\hat{T}|^{1/2}} (\int_{\hat{T}} \hat{\sigma}^{\mu-2}(|\hat{\boldsymbol{p}}_1|^2 + \sum_{\hat{F}\in\partial\hat{T}} |\widehat{\boldsymbol{v}_F \cdot \boldsymbol{n}_F}|^2) d\hat{\boldsymbol{x}})^{1/2} \\
&\le C_9 |T|^{1/2} (\int_{\hat{T}} \hat{\sigma}^{\mu-2}(\sum_{i=1}^{d+1} |\hat{\boldsymbol{v}}(\hat{\boldsymbol{a}}_i)|^2 + \sum_{\hat{F}\in\partial\hat{T}} |\widehat{\boldsymbol{v}_F \cdot \boldsymbol{n}_F}|^2) d\hat{\boldsymbol{x}})^{1/2} \\
&\le C_{10} |T|^{1/2} \|\hat{\sigma}^{\mu/2-1}\hat{\boldsymbol{v}}\|_{L^2(\hat{T})} \le C_{11} \|\sigma^{\mu/2-1}\boldsymbol{v}_h\|_{L^2(T)}.
\end{aligned}$$

This proves the theorem. ∎

Finally, r_h is the orthogonal projection on $I\!P_0$ in each T:

$$r_h(q)|_T = \frac{1}{|T|} \int_T q(\boldsymbol{x}) d\boldsymbol{x} \quad \forall T \in \mathcal{T}_h. \tag{6.30}$$

By definition, r_h preserves the mean-value:

$$\int_\Omega r_h(q)(\boldsymbol{x}) d\boldsymbol{x} = \int_\Omega q(\boldsymbol{x}) d\boldsymbol{x}.$$

7 Estimates for the pressure

This section is devoted to estimating the fourth term of (0.21):

$$\int_\Omega (Q - Q_h) \mathrm{div}\,(\bar{P}_h(\boldsymbol{\psi}))\, d\boldsymbol{x},$$

in terms of $\sigma^{\mu/2}\nabla(\boldsymbol{G}-\boldsymbol{G}_h)$. Here we assume that the finite elements satisfy (6.1). Since $\bar{P}_h$ satisfies (3.21), we can write

$$\begin{aligned}\int_\Omega (Q-Q_h)\mathrm{div}\,(\bar{P}_h(\boldsymbol{\psi}))\,d\boldsymbol{x} &= \int_\Omega (Q-r_h(Q))\mathrm{div}\,(\bar{P}_h(\boldsymbol{\psi}))\,d\boldsymbol{x} + \int_\Omega (r_h(Q)-Q_h)\mathrm{div}\,\boldsymbol{\psi}\,d\boldsymbol{x}\\ &= \int_\Omega (Q-r_h(Q))\mathrm{div}\,(\bar{P}_h(\boldsymbol{\psi})-\boldsymbol{\psi})\,d\boldsymbol{x} + \int_\Omega (Q-r_h(Q))\mathrm{div}\,\boldsymbol{\psi}\,d\boldsymbol{x}\\ &+ \int_\Omega (r_h(Q)-Q_h)\mathrm{div}\,\boldsymbol{\psi}\,d\boldsymbol{x}\,.\end{aligned} \tag{7.1}$$

Lemma 7.1 *Under the assumptions of Theorem 4.2 and Theorem 5.1 with $\varepsilon=0$, we have*

$$|\int_\Omega (Q-r_h(Q))\mathrm{div}\,\boldsymbol{\psi}\,d\boldsymbol{x}| \le C_1\kappa^{\mu+1/2}h^\lambda + \frac{C_2}{\sqrt{\kappa}}\|\sigma^{\mu/2}\nabla(\boldsymbol{G}-\boldsymbol{G}_h)\|^2_{L^2(\Omega)}\,. \tag{7.2}$$

Proof. In view of (3.28), we write

$$\begin{aligned}|\int_\Omega (Q-r_h(Q))\mathrm{div}\,\boldsymbol{\psi}\,d\boldsymbol{x}| &\le \|\sigma^{\mu/2}(Q-r_h(Q))\|_{L^2(\Omega)}\|\sigma^{-\mu/2}\mathrm{div}\,\boldsymbol{\psi}\|_{L^2(\Omega)}\\ &\le C_1\kappa^{\mu/2}h^{\lambda/2}\|\sigma^{-\mu/2}\mathrm{div}\,\boldsymbol{\psi}\|_{L^2(\Omega)}\,,\end{aligned} \tag{7.3}$$

where $\psi=\sigma^\mu(P_h(\boldsymbol{G})-\boldsymbol{G}_h))$. Next, expanding $\boldsymbol{\psi}$ and applying (2.1), we obtain

$$\begin{aligned}\|\sigma^{-\mu/2}\mathrm{div}\,\boldsymbol{\psi}\|_{L^2(\Omega)} &\le \|\sigma^{\mu/2}\mathrm{div}(P_h(\boldsymbol{G})-\boldsymbol{G}_h)\|_{L^2(\Omega)} + \mu\|\sigma^{\mu/2-1}(P_h(\boldsymbol{G})-\boldsymbol{G}_h)\|_{L^2(\Omega)}\\ &\le \sqrt{d}(\|\sigma^{\mu/2}\nabla(P_h(\boldsymbol{G})-\boldsymbol{G})\|_{L^2(\Omega)} + \|\sigma^{\mu/2}\nabla(\boldsymbol{G}-\boldsymbol{G}_h)\|_{L^2(\Omega)})\\ &+ \mu(\|\sigma^{\mu/2-1}(P_h(\boldsymbol{G})-\boldsymbol{G})\|_{L^2(\Omega)} + \|\sigma^{\mu/2-1}(\boldsymbol{G}-\boldsymbol{G}_h)\|_{L^2(\Omega)})\,.\end{aligned}$$

Therefore, applying (3.28) and (3.29) and considering that $\kappa>1$, we get

$$\|\sigma^{-\mu/2}\mathrm{div}\,\boldsymbol{\psi}\|_{L^2(\Omega)} \le C_2\kappa^{\mu/2}h^{\lambda/2} + \sqrt{d}\|\sigma^{\mu/2}\nabla(\boldsymbol{G}-\boldsymbol{G}_h)\|_{L^2(\Omega)} + \mu\|\sigma^{\mu/2-1}(\boldsymbol{G}-\boldsymbol{G}_h)\|_{L^2(\Omega)}\,.$$

Then Corollary 5.3 gives

$$\|\sigma^{-\mu/2}\mathrm{div}\,\boldsymbol{\psi}\|_{L^2(\Omega)} \le C_3\kappa^{\mu/2}h^{\lambda/2} + (\sqrt{d}+\frac{\mu}{\sqrt{\kappa}}(1+2C_0^2)^{1/2})\|\sigma^{\mu/2}\nabla(\boldsymbol{G}-\boldsymbol{G}_h)\|_{L^2(\Omega)}\,.$$

When substituted into (7.3), we recover (7.2). ∎

Lemma 7.2 *We retain the assumptions and notation of Lemma 7.1; then*

$$|\int_\Omega (Q-r_h(Q))\mathrm{div}(\bar{P}_h(\boldsymbol{\psi})-\boldsymbol{\psi})\,d\boldsymbol{x}| \le C_1\kappa^{\mu-1/2}h^\lambda + \frac{C_2}{\sqrt{\kappa}}\|\sigma^{\mu/2}\nabla(\boldsymbol{G}-\boldsymbol{G}_h)\|^2_{L^2(\Omega)}. \tag{7.4}$$

Proof. By virtue of (6.1), (3.28) and (3.29), we can write

$$\begin{aligned}|\int_\Omega (Q-r_h(Q))\mathrm{div}(\bar{P}_h(\boldsymbol{\psi})-\boldsymbol{\psi})\,d\boldsymbol{x}| &\le C_1\|\sigma^{\mu/2}(Q-r_h(Q))\|_{L^2(\Omega)}\|\sigma^{\mu/2-1}(P_h(\boldsymbol{G})-\boldsymbol{G}_h)\|_{L^2(\Omega)}\\ &\le C_2\kappa^{\mu/2}h^{\lambda/2}(\|\sigma^{\mu/2-1}(P_h(\boldsymbol{G})-\boldsymbol{G})\|_{L^2(\Omega)} + \|\sigma^{\mu/2-1}(\boldsymbol{G}-\boldsymbol{G}_h)\|_{L^2(\Omega)})\\ &\le C_2\kappa^{\mu/2}h^{\lambda/2}(C_3\kappa^{\mu/2-1}h^{\lambda/2} + \|\sigma^{\mu/2-1}(\boldsymbol{G}-\boldsymbol{G}_h)\|_{L^2(\Omega)})\,.\end{aligned}$$

Then (7.4) follows by applying Corollary 5.3 to this inequality. ∎

It remains to study the last term in (7.1), which we expand as follows:

$$\begin{aligned}\int_\Omega (r_h(Q)-Q_h)\mathrm{div}\,\boldsymbol{\psi}\,d\boldsymbol{x} &= \int_\Omega \sigma^\mu (r_h(Q)-Q_h)\mathrm{div}(\boldsymbol{G}_h - P_h(\boldsymbol{G}))\,d\boldsymbol{x} \\ &+ \int_\Omega (r_h(Q)-Q_h)\nabla\,\sigma^\mu \cdot (\boldsymbol{G}_h - P_h(\boldsymbol{G}))\,d\boldsymbol{x}\,. \end{aligned} \tag{7.5}$$

For the first term in (7.5), we introduce an auxiliary approximation operator $\bar{r}_h$ that satisfies the analogue of the super-approximation result (6.1): if $q_h \in \bar{M}_h$ and $\zeta = \sigma^\mu q_h$, then

$$\|\sigma^{-\mu/2}(\zeta - \bar{r}_h(\zeta))\|_{L^2(\Omega)} \le C\,h\|\sigma^{\mu/2-1}q_h\|_{L^2(\Omega)}\,. \tag{7.6}$$

In the examples of Section 6, $\bar{r}_h$ coincides with r_h for the Bernardi-Raugel element (cf.(6.30)) and $\bar{r}_h = I_h$, the $\mathbb{P}_{k-1}$ Lagrange interpolant for the Taylor-Hood $\mathbb{P}_k$–$\mathbb{P}_{k-1}$ element and the $\mathbb{P}_1$ Lagrange interpolant for the mini-element.

Lemma 7.3 *Let $\mathcal{T}_h$ satisfy (1.3); then the operator r_h defined by (6.30) satisfies (7.6).*

Proof. Since r_h preserves the constant functions in each T, we have

$$\|\sigma^{-\mu/2}(\zeta - r_h(\zeta))\|_{L^2(\Omega)} \le \sup_{\boldsymbol{x}\in T}\sigma^{-\mu/2}(\boldsymbol{x})C_1 h_T\|\nabla\,\zeta\|_{L^2(T)}\,.$$

But the degree of q_h implies that

$$\nabla\,\zeta = \nabla\,\sigma^\mu q_h\,.$$

Hence

$$\|\nabla\,\zeta\|_{L^2(T)} \le \mu\sup_{\boldsymbol{x}\in T}\sigma^{\mu/2}(\boldsymbol{x})\|\sigma^{\mu/2-1}q_h\|_{L^2(T)}\,,$$

and (7.6) follows from Lemma 2.1. ∎

Lemma 7.4 *Let $\mathcal{T}_h$ satisfy (1.3); then the $\mathbb{P}_{k-1}$ Lagrange interpolant satisfies (7.6).*

We skip the proof because it can be found in similar works on the Laplace equation, for instance [7].

Proposition 7.5 *Under the assumptions of Theorem 4.2, we have*

$$\Big|\int_\Omega \sigma^\mu (r_h(Q)-Q_h)\mathrm{div}(\boldsymbol{G}_h - P_h(\boldsymbol{G}))\,d\boldsymbol{x}\Big| \le C_1\kappa^{\mu-1}h^\lambda + \frac{C_2}{\kappa}\|\sigma^{\mu/2}\nabla(\boldsymbol{G}-\boldsymbol{G}_h)\|^2_{L^2(\Omega)}. \tag{7.7}$$

Proof. Set $\zeta = \sigma^\mu(r_h(Q)-Q_h)$. As $\boldsymbol{G}_h - P_h(\boldsymbol{G}) \in V_h$, we have:

$$\int_\Omega \sigma^\mu (r_h(Q)-Q_h)\mathrm{div}(\boldsymbol{G}_h - P_h(\boldsymbol{G}))\,d\boldsymbol{x} = \int_\Omega (\zeta - \bar{r}_h(\zeta))\mathrm{div}(\boldsymbol{G}_h - P_h(\boldsymbol{G}))\,d\boldsymbol{x}\,.$$

Therefore (7.6) implies that

$$\begin{aligned}\Big|\int_\Omega \sigma^\mu (r_h(Q)-Q_h)\mathrm{div}(\boldsymbol{G}_h - P_h(\boldsymbol{G}))\,d\boldsymbol{x}\Big| &\le \sqrt{d}\|\sigma^{\mu/2}\nabla(\boldsymbol{G}_h - P_h(\boldsymbol{G}))\|_{L^2(\Omega)}\|\sigma^{-\mu/2}(\zeta - \bar{r}_h(\zeta))\|_{L^2(\Omega)} \\ &\le C_1 h\|\sigma^{\mu/2}\nabla(\boldsymbol{G}_h - P_h(\boldsymbol{G}))\|_{L^2(\Omega)}\|\sigma^{\mu/2-1}(r_h(Q)-Q_h)\|_{L^2(\Omega)}\,.\end{aligned}$$

Thus, applying Theorem 4.2 with $\alpha = \mu - 2$ and using (3.27), we obtain

$$|\int_\Omega \sigma^\mu (r_h(Q) - Q_h)\mathrm{div}(\boldsymbol{G}_h - P_h(\boldsymbol{G}))\, d\boldsymbol{x}| \le \frac{C_2}{\kappa}(\|\sigma^{\mu/2}\nabla(\boldsymbol{G} - \boldsymbol{G}_h)\|_{L^2(\Omega)} + C_3\kappa^{\mu/2}h^{\lambda/2}))$$
$$\times (\|\sigma^{\mu/2}\nabla(\boldsymbol{G}_h - \boldsymbol{G})\|_{L^2(\Omega)} + \|\sigma^{\mu/2}\nabla(\boldsymbol{G} - P_h(\boldsymbol{G}))\|_{L^2(\Omega)}) \,.$$

Then (7.7) follows by applying (3.28). ∎

In order to bound the second term in (7.5), we choose in Section 5, $\varepsilon = \lambda + \gamma$ for some small number $\gamma > 0$ and we assume that $\partial\Omega$ is such that (5.1) holds for some real number $r > d$. Then, if for instance, we take

$$\gamma = \frac{\lambda}{2}\,,$$

condition (5.2) reads

$$2\lambda < 1 - \frac{d}{r}\,. \tag{7.8}$$

Proposition 7.6 *We suppose that $\mathcal{T}_h$ satisfies (2.5) and (5.1) holds for some real number $r > d$. Let $\lambda > 0$ satisfy (7.8). Then*

$$|\int_\Omega (r_h(Q) - Q_h)\nabla\,\sigma^\mu \cdot (\boldsymbol{G}_h - P_h(\boldsymbol{G}))\, d\boldsymbol{x}| \le C_1\kappa^{\mu-1/2}h^\lambda + \frac{C_2}{\sqrt{\kappa}}\|\sigma^{\mu/2}\nabla(\boldsymbol{G} - \boldsymbol{G}_h)\|^2_{L^2(\Omega)}. \tag{7.9}$$

Proof. The proof is written for positive arbitrary λ and γ satisfying $3\lambda/2 + \gamma < 1 - d/r$; in particular it is valid for λ satisfying (7.8). From (2.1), we have

$$\begin{aligned}|\int_\Omega (r_h(Q) - Q_h)\nabla\,\sigma^\mu \cdot (\boldsymbol{G}_h - P_h(\boldsymbol{G}))\, d\boldsymbol{x}| &\le \mu\int_\Omega \sigma^{\mu-1}|r_h(Q) - Q_h|(|\boldsymbol{G}_h - \boldsymbol{G}| + |\boldsymbol{G} - P_h(\boldsymbol{G})|)d\boldsymbol{x}\\ &\le \mu\|\sigma^{(\mu-\lambda-\gamma)/2}(r_h(Q) - Q_h)\|_{L^2(\Omega)}\|\sigma^{(\mu+\lambda+\gamma)/2-1}(\boldsymbol{G}_h - \boldsymbol{G})\|_{L^2(\Omega)}\\ &+ \mu\|\sigma^{\mu/2-1}(r_h(Q) - Q_h)\|_{L^2(\Omega)}\|\sigma^{\mu/2}(\boldsymbol{G} - P_h(\boldsymbol{G}))\|_{L^2(\Omega)}\,.\end{aligned} \tag{7.10}$$

For the first term in the above right-hand side, we apply Theorem 4.2 with $\alpha = \mu - \lambda - \gamma = d - \gamma < d$ and Corollary 5.4 with $\varepsilon = \lambda + \gamma$. These two results give

$$\begin{aligned}&\|\sigma^{(\mu-\lambda-\gamma)/2}(r_h(Q) - Q_h)\|_{L^2(\Omega)}\|\sigma^{(\mu+\lambda+\gamma)/2-1}(\boldsymbol{G}_h - \boldsymbol{G})\|_{L^2(\Omega)}\\ &\quad\le \frac{C_1}{\sqrt{\kappa}}(C_2\|\sigma^{\mu/2}\nabla(\boldsymbol{G} - \boldsymbol{G}_h)\|^2_{L^2(\Omega)} + C_3\kappa^\mu h^\lambda)^{1/2} \times (\|\sigma^{\mu/2}\nabla(\boldsymbol{G} - \boldsymbol{G}_h)\|_{L^2(\Omega)} + C_4\kappa^{\mu/2}h^{\lambda/2})\\ &\quad\le \frac{1}{\sqrt{\kappa}}(C_5\|\sigma^{\mu/2}\nabla(\boldsymbol{G} - \boldsymbol{G}_h)\|^2_{L^2(\Omega)} + C_6\kappa^\mu h^\lambda)\,.\end{aligned}$$

For the second term in the right-hand side of (7.10), we apply (3.29) to the second factor and Theorem 4.2 with $\alpha = \mu - 2$ to the first factor. With (3.27), these two results give:

$$\|\sigma^{\mu/2-1}(r_h(Q) - Q_h)\|_{L^2(\Omega)}\|\sigma^{\mu/2}(\boldsymbol{G} - P_h(\boldsymbol{G}))\|_{L^2(\Omega)} \le C_8\kappa^{\mu/2-1}h^{\lambda/2}\|\sigma^{\mu/2}\nabla(\boldsymbol{G} - \boldsymbol{G}_h)\|_{L^2(\Omega)} + C_9\kappa^{\mu-1}h^\lambda\,,$$

whence (7.9). ∎

Collecting (7.1), (7.2), (7.4), (7.7) and (7.9), we derive the estimate for the pressure.

Theorem 7.7 *We suppose that $\mathcal{T}_h$ satisfies (2.5) and (5.1) holds for some real number $r > d$. Let $\lambda > 0$ satisfy (7.8). Then*

$$|\int_\Omega (Q - Q_h)\mathrm{div}\,(\bar{P}_h(\boldsymbol{\psi}))\,d\boldsymbol{x}| \le C_1\kappa^{\mu+1/2}h^\lambda + \frac{C_2}{\sqrt{\kappa}}\|\sigma^{\mu/2}\nabla(\boldsymbol{G} - \boldsymbol{G}_h)\|^2_{L^2(\Omega)}\,. \tag{7.11}$$

8 Final estimates

8.1 Velocity estimates

Collecting the results of the previous sections, we obtain the estimate (0.19). We recall that R is the radius of the fixed ball $B(\boldsymbol{x}_0; R)$ containing Ω (cf. Remark 1.4).

Theorem 8.1 *Assume that $\mathcal{T}_h$ satisfies (2.5) and (5.1) holds for some real number $r > d$. Let $\mu = d + \lambda$ where $\lambda > 0$ satisfies (7.8). Then there exists a number $\kappa_1 > 1$ such that for all $\kappa \ge \kappa_1$ and for all mesh size $h > 0$ such that*

$$\kappa\, h \le R\,,$$

we have

$$\|\sigma^{\mu/2}\nabla(\boldsymbol{G} - \boldsymbol{G}_h)\|_{L^2(\Omega)} \le C\kappa^{\mu/2+1/4}h^{\lambda/2}\,. \tag{8.1}$$

Proof. From (0.21), we obtain

$$\begin{aligned}\|\sigma^{\mu/2}\nabla(\boldsymbol{G} - \boldsymbol{G}_h)\|^2_{L^2(\Omega)} \le{}& \|\sigma^{\mu/2}\nabla(\boldsymbol{G} - \boldsymbol{G}_h)\|_{L^2(\Omega)}\big(\|\sigma^{\mu/2}\nabla(\boldsymbol{G} - P_h(\boldsymbol{G}))\|_{L^2(\Omega)}\\ &+ \mu(\|\sigma^{\mu/2-1}(\boldsymbol{G} - P_h(\boldsymbol{G}))\|_{L^2(\Omega)} + \|\sigma^{\mu/2-1}(\boldsymbol{G} - \boldsymbol{G}_h)\|_{L^2(\Omega)})\\ &+ \|\sigma^{-\mu/2}\nabla(\boldsymbol{\psi} - \bar{P}_h(\boldsymbol{\psi})\|_{L^2(\Omega)}\big) + |\int_\Omega (Q - Q_h)\mathrm{div}\,(\bar{P}_h(\boldsymbol{\psi}))\,d\boldsymbol{x}|\,.\end{aligned}$$

First, applying Theorem 3.11, (6.1) and Theorem 7.7, this reduces to

$$\begin{aligned}\|\sigma^{\mu/2}\nabla(\boldsymbol{G} - \boldsymbol{G}_h)\|^2_{L^2(\Omega)} \le{}& \|\sigma^{\mu/2}\nabla(\boldsymbol{G} - \boldsymbol{G}_h)\|_{L^2(\Omega)}\big(C_1\kappa^{\mu/2}h^{\lambda/2} + C_2\kappa^{\mu/2-1}h^{\lambda/2}\\ &+ C_3\|\sigma^{\mu/2-1}(\boldsymbol{G} - \boldsymbol{G}_h)\|_{L^2(\Omega)}\big) + C_4\kappa^{\mu+1/2}h^\lambda + \frac{C_5}{\sqrt{\kappa}}\|\sigma^{\mu/2}\nabla(\boldsymbol{G} - \boldsymbol{G}_h)\|^2_{L^2(\Omega)}\,.\end{aligned}$$

Next, applying Corollary 5.3, we obtain

$$\begin{aligned}\|\sigma^{\mu/2}\nabla(\boldsymbol{G} - \boldsymbol{G}_h)\|^2_{L^2(\Omega)} \le{}& \|\sigma^{\mu/2}\nabla(\boldsymbol{G} - \boldsymbol{G}_h)\|_{L^2(\Omega)}\big(C_1\kappa^{\mu/2}h^{\lambda/2} + C_2'\kappa^{\mu/2-1/2}h^{\lambda/2}\big)\\ &+ \frac{C_3'}{\sqrt{\kappa}}\|\sigma^{\mu/2}\nabla(\boldsymbol{G} - \boldsymbol{G}_h)\|^2_{L^2(\Omega)} + C_4\kappa^{\mu+1/2}h^\lambda\,.\end{aligned} \tag{8.2}$$

Finally, let us choose κ_1 such that for instance

$$\frac{C_3'}{\sqrt{\kappa_1}} = \frac{1}{2}\,. \tag{8.3}$$

Then for all $\kappa \ge \kappa_1$ and all $h > 0$ such that $\kappa\, h \le R$, (8.2) implies (8.1). ∎

Combining Theorem 8.1 with Lemma 1.2, (0.17) and Lemma 1.3, we derive the main result of this work for the velocity.

Theorem 8.2 *Under the assumptions of Theorem 8.1 and provided the solution $(\boldsymbol{u}, p)$ of the Stokes problem (0.1), (0.2) belongs to $W^{1,\infty}(\Omega)^d \times L^\infty(\Omega)$, we have*

$$\|\nabla\,\boldsymbol{u}_h\|_{L^\infty(\Omega)} \le C\left(\|\nabla\,\boldsymbol{u}\|_{L^\infty(\Omega)} + \|p\|_{L^\infty(\Omega)}\right)\,, \tag{8.4}$$

with a constant C independent of h, $\boldsymbol{u}$ and p.

8.2 Pressure estimates

We proceed by duality because an L^∞ estimate for the pressure cannot be obtained directly from the previous results, since the inf-sup condition is usually not valid in L^∞. Let $\boldsymbol{x}_M$ be a point in $\bar{\Omega}$ where $|p_h(\boldsymbol{x})|$ attains its maximum, let δ_M be the function constructed in Lemma 1.1 with $\varphi_h = p_h$ and let $(\boldsymbol{G}, Q) \in H^1_0(\Omega)^d \times L^2_0(\Omega)$ be the solution of

$$-\Delta\,\boldsymbol{G} + \nabla\,Q = 0\ ,\ \operatorname{div}\boldsymbol{G} = \delta_M - B\,, \tag{8.5}$$

where B is a fixed function of $\mathcal{D}(\Omega)$ such that $\int_\Omega B(\boldsymbol{x})\,d\boldsymbol{x} = 1$. By virtue of (1.4), $\delta_M - B$ belongs to $L^2_0(\Omega)$ and Problem (8.5) has a unique solution. Furthermore, since $\delta_M - B$ belongs to $\mathcal{D}(\Omega)$, in view of Theorem 0.4, there exists a function $\boldsymbol{v}$ in $H^2_0(\Omega)^d$ such that

$$\operatorname{div}\boldsymbol{v} = \delta_M - B\ ,\ \|\boldsymbol{v}\|_{H^2(\Omega)} \le C\|\delta_M - B\|_{H^1(\Omega)}\,. \tag{8.6}$$

Subtracting $\boldsymbol{v}$ from (8.5), we see that $\boldsymbol{G}-\boldsymbol{v}$ solves a homogeneous Stokes problem with data $\Delta\,\boldsymbol{v} \in L^2(\Omega)^d$. Thus, we deduce the regularity of $\boldsymbol{G}$ solely from the angles of $\partial\Omega$.

Then, we define $\boldsymbol{G}_h \in X_h$, the Stokes projection of $\boldsymbol{G}$, and its associated pressure $Q_h \in M_h$ by

$$\int_\Omega \nabla(\boldsymbol{G}_h - \boldsymbol{G}) : \nabla\,\boldsymbol{v}_h\,d\boldsymbol{x} + \int_\Omega (Q - Q_h)\operatorname{div}\boldsymbol{v}_h\,d\boldsymbol{x} = 0 \quad \forall \boldsymbol{v}_h \in X_h\,, \tag{8.7}$$

$$\int_\Omega q_h \operatorname{div}(\boldsymbol{G}_h - G)\,d\boldsymbol{x} = 0 \quad \forall q_h \in M_h\,. \tag{8.8}$$

As in [16], we derive the following result:

Lemma 8.3 *Let the operator r_h be defined as in the previous sections. Then*

$$\|p_h\|_{L^\infty(\Omega)} \le C\big(\|p\|_{L^\infty(\Omega)} + \|\nabla\boldsymbol{u}\|_{L^\infty(\Omega)}\big)\big(\|\nabla(\boldsymbol{G}-\boldsymbol{G}_h)\|_{L^1(\Omega)} + \|Q - r_h(Q)\|_{L^1(\Omega)}\big). \tag{8.9}$$

Proof. In view of (1.5), we have

$$\|p_h\|_{L^\infty(\Omega)} = \int_\Omega (\delta_M - B)p_h d\boldsymbol{x} + \int_\Omega B\,p_h d\boldsymbol{x}\,.$$

Then, applying (8.5), (8.7), (8.8), (0.1) and (0.2), we deduce:

$$\begin{aligned}\|p_h\|_{L^\infty(\Omega)} = &\int_\Omega \nabla(\boldsymbol{G}_h - \boldsymbol{G}) : \nabla(\boldsymbol{u}_h - \boldsymbol{u})d\boldsymbol{x} + \int_\Omega (Q - r_h(Q))\operatorname{div}(\boldsymbol{u}_h - \boldsymbol{u})\,d\boldsymbol{x}\\ &+ \int_\Omega p\ \operatorname{div}(\boldsymbol{G}_h - \boldsymbol{G})\,d\boldsymbol{x} + \int_\Omega \delta_M\ p\,d\boldsymbol{x}\ + \int_\Omega B(p_h - p)\,d\boldsymbol{x}\,.\end{aligned}$$

As B is fixed, this together with (0.5) implies that

$$\int_\Omega B(p_h - p)\,d\boldsymbol{x} \le \|B\|_{L^2(\Omega)}\|p_h - p\|_{L^2(\Omega)} \le C(\|p\|_{L^2(\Omega)} + \|\nabla\boldsymbol{u}\|_{L^2(\Omega)}).$$

Inserting this back into the previous estimate and making use of (8.4) yields the assertion. ∎

To proceed further, we need a uniform estimate for $\nabla(\boldsymbol{G}_h - \boldsymbol{G})$ and $r_h(Q) - Q$ in $L^1(\Omega)$; to be specific, we shall prove the weighted estimates

$$\|\sigma^{\mu/2}\nabla(\boldsymbol{G} - \boldsymbol{G}_h)\|_{L^2(\Omega)} \le C\kappa^{\mu/2+1/4}h^{\lambda/2}\,, \tag{8.10}$$

$$\|\sigma^{\mu/2}(r_h(Q) - Q)\|_{L^2(\Omega)} \le C\kappa^{\mu/2+1/4}h^{\lambda/2}\,, \tag{8.11}$$

with $\mu = d + \lambda$, $\lambda > 0$ and C independent of κ and h. Since the equations (8.5) defining $(\boldsymbol{G}, Q)$ are similar to (0.12) and (0.13), these two estimates are an easy variant of (8.1) and (3.28). Therefore, we shall only examine the points where the proofs differ.

First of all, as (1.16) is valid here, (0.21) is unchanged and we must revisit the weighted interpolation errors of Section 3. Under the hypotheses of Proposition 3.1, the estimate (3.1) simplifies to

$$\|\sigma^{\mu/2-1} Q\|_{L^2(\Omega)} \leq C \, \|\sigma^{\mu/2-1} \nabla \boldsymbol{G}\|_{L^2(\Omega)} \,. \tag{8.12}$$

Similarly, (3.5) simplifies to

$$\|\sigma^{\mu/2-1} \nabla \boldsymbol{G}\|_{L^2(\Omega)} \leq C \, \|\sigma^{\mu/2-2} \boldsymbol{G}\|_{L^2(\Omega)} \,. \tag{8.13}$$

The statement of the duality Theorem 3.3 is unchanged. Indeed, we use the same dual problem (3.10), (3.11) still holds, but we also need here the analogue estimate for the dual pressure r:

$$\|r\|_{W^{1,2s/(2s-1)}(\Omega)} \leq C_1 \left(\int_\Omega |\boldsymbol{G}|^{2s} d\boldsymbol{x} \right)^{(2s-1)/2s} .$$

Then (3.12) becomes

$$\int_\Omega |\boldsymbol{G}|^{2s} d\boldsymbol{x} = - \int_\Omega r \operatorname{div} \boldsymbol{G} \, d\boldsymbol{x} = - \int_\Omega (\delta_M - B) r d\boldsymbol{x} \,.$$

Hence, for any $t' > 1$ such that $W^{1,2s/(2s-1)}(\Omega) \subset L^{t'}(\Omega)$,

$$\|\boldsymbol{G}\|^{2s}_{L^{2s}(\Omega)} \leq \|\delta_M - B\|_{L^t(\Omega)} \|r\|_{L^{t'}(\Omega)} \,, \quad \frac{1}{t} + \frac{1}{t'} = 1 \,,$$

and since $\|B\|_{L^t(\Omega)}$ is a fixed constant that depends only on t, the remainder of the proof is unchanged. From Theorem 3.3, (8.12) and (8.13), we deduce the analogue of (3.14) with the same exponent for h and a smaller exponent for κ:

$$\|\sigma^{\mu/2-1} \nabla \boldsymbol{G}\|_{L^2(\Omega)} + \|\sigma^{\mu/2-1} Q\|_{L^2(\Omega)} \leq C_t \kappa^{d(1-1/t)+\lambda/2-1} h^{\lambda/2-1} \,. \tag{8.14}$$

Similarly, the statement of Theorem 3.6 is unchanged. Indeed, (3.18) is replaced by

$$-\Delta(\sigma^{\mu/2} \boldsymbol{G}) + \nabla(\sigma^{\mu/2} Q) = -2(\nabla(\sigma^{\mu/2}) \cdot \nabla) \boldsymbol{G} - \Delta(\sigma^{\mu/2}) \boldsymbol{G} + \nabla(\sigma^{\mu/2}) Q \in L^2(\Omega)^d \,,$$
$$\operatorname{div}(\sigma^{\mu/2} \boldsymbol{G}) = \sigma^{\mu/2}(\delta_M - B) + \nabla(\sigma^{\mu/2}) \cdot \boldsymbol{G} \in H^1_0(\Omega) \,.$$

Again, as $\sigma^{\mu/2} \boldsymbol{G}$ vanishes on $\partial\Omega$, we have that $(\sigma^{\mu/2}(\delta_M - B) + \nabla(\sigma^{\mu/2}) \cdot \boldsymbol{G})$ belongs to $H^1_0(\Omega) \cap L^2_0(\Omega)$ and applying Theorem 0.4 there exists $\boldsymbol{v}$ in $H^2_0(\Omega)^d$ such that

$$\operatorname{div} \boldsymbol{v} = \sigma^{\mu/2}(\delta_M - B) + \nabla(\sigma^{\mu/2}) \cdot \boldsymbol{G} \,,$$
$$\|\boldsymbol{v}\|_{H^2(\Omega)} \leq C_1 \|\sigma^{\mu/2}(\delta_M - B) + \nabla(\sigma^{\mu/2}) \cdot \boldsymbol{G}\|_{H^1(\Omega)} \,.$$

Then (2.6) and (2.7) and the fact that B is smooth and fixed yield

$$\|\boldsymbol{v}\|_{H^2(\Omega)} \leq C_1 \|\nabla(\sigma^{\mu/2}) \cdot \boldsymbol{G}\|_{H^1(\Omega)} + C_2 \kappa^{\mu/2} h^{\lambda/2-1} + C_3 \kappa^{\mu/2-1} h^{\lambda/2-1} \,,$$

and we recover the statement of Theorem 3.6. As a consequence, the weighted error estimates of Theorem 3.11 are valid here.

Then the discrete inf-sup condition of Theorem 4.2 holds. Finally, it is easy to check that the general duality argument of Section 5 is unchanged because it involves the difference $\boldsymbol{G} - \boldsymbol{G}_h$ whose divergence is orthogonal to the functions of M_h. The same is true for the pressure estimates of Section 7. Hence, when all the estimates above are collected in (0.21), they yield the same estimate as (8.1) with possibly another constant, still independent of h and κ. With Lemma 8.3, this proves the following pressure estimate.

Theorem 8.4 *Under the assumptions of Theorem 8.2, there exists a constant $C > 0$ independent of $h, \boldsymbol{u}$ and p such that*

$$\|p_h\|_{L^\infty(\Omega)} \leq C \big(\|\nabla \boldsymbol{u}\|_{L^\infty(\Omega)} + \|p\|_{L^\infty(\Omega)} \big) . \tag{8.15}$$

8.3 Optimal error estimates

Upon taking $\boldsymbol{u} - \boldsymbol{v}_h$ and $p - q_h$, with arbitrary $\boldsymbol{v}_h \in X_h$ and $q_h \in M_h$, instead of $\boldsymbol{u}$ and p in the stability bounds (8.4) and (8.15), and realizing that the Stokes projection is invariant on the discrete product space $X_h \times M_h$, we readily derive the following log-free error estimate in the maximum norm:

$$\|\nabla(\boldsymbol{u} - \boldsymbol{u}_h)\|_{L^\infty(\Omega)} + \|p - p_h\|_{L^\infty(\Omega)} \leq C \inf_{(\boldsymbol{v}_h, q_h) \in X_h \times M_h} \Big(\|\nabla(\boldsymbol{u} - \boldsymbol{v}_h)\|_{L^\infty(\Omega)} + \|p - q_h\|_{L^\infty(\Omega)} \Big).$$

We stress that the regularity requirements on the domain Ω are the minimal conditions that suffice to guarantee that $(\boldsymbol{u}, p) \in W^{1,\infty}(\Omega)^d \times L^\infty(\Omega)$, and consequently that this error estimate makes sense.

References

[1] R. A. Adams, *Sobolev Spaces*, Academic Press, New York, NY, 1975.

[2] C. Amrouche and V. Girault, *Decomposition of vector spaces and application to the Stokes problem in arbitrary dimensions,* Czech. Math. Journal **44**, (1994), pp. 109–140.

[3] D. Arnold, F. Brezzi and M. Fortin, *A stable finite element for the Stokes equations*, Calcolo **21**, 4 (1984), pp. 337–344.

[4] D. Arnold and X. Liu, *Local error estimates for finite element discretizations of the Stokes equations*, M2AN **29**, (1995), pp. 367–389.

[5] C. Bernardi and G. Raugel, *Analysis of some finite elements for the Stokes problem*, Math. Comp. **44**, 169 (1985), pp. 71–79.

[6] J. Boland and R. Nicolaides, *Stablility of finite elements under divergence constraints*, SIAM J. Numer. Anal. **20**, 4 (1983), pp. 722–731.

[7] S. Brenner and L. R. Scott, *The Mathematical Theory of Finite Element Methods*, Springer-Verlag, New York, NY, 1994.

[8] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, Springer-Verlag, New York, 1991.

[9] H. Chen, *Pointwise error estimates for finite element solutions of the Stokes problem*, preprint (2002).

[10] P. G. Ciarlet, *Basic error estimates for elliptic problems - Finite Element Methods, Part 1*, Handbook of Numerical Analysis, P. G. Ciarlet and J. L. Lions, eds., North-Holland, Amsterdam, 1991.

[11] P. G. Ciarlet and P.-A. Raviart, *Maximum principle and uniform convergence for the finite element method*, Comp. Meth. Appl. Mech. Eng. **2**, (1973), pp. 17–31.

[12] P. Ciarlet Jr, and V. Girault, *Condition inf-sup pour l'élément fini de Taylor-Hood P_2-iso-P_1, 3-D; applications aux équations de Maxwell*, CRAS, Ser. I, **335**, (2002), pp. 827–832.

[13] M. Dauge, *Stationary Stokes and Navier-Stokes systems on two or three-dimensional domains with corners*, SIAM J. Math. Anal. **20**, 1 (1989), pp. 74–97.

[14] A. Demlow, *Localized pointwise error estimates for mixed finite element methods*, preprint (2003).

[15] R. Durán and A. Muschietti, *An explicit right inverse of the divergence operator which is continuous in weighted norms*, Studia Mathematica **148**, (2001), pp. 207–219.

[16] R. Durán, R. H. Nochetto and J. Wang, *Sharp maximum norm error estimates for finite element approximations of the Stokes problem in* $2-D$, Math. Comp. **51**, 184 (1988), pp. 1177–1192.

[17] M. Fortin, *An analysis of the convergence of mixed finite element methods*, RAIRO Anal. Numér. **11**, R3 (1977), pp. 341–354.

[18] L. Gastaldi and R. Nochetto, *Quasi-optimal pointwise error estimates for the Reissner-Mindlin plate*, SIAM J. Numer. Anal. **28**, 2 (1991), pp. 363–377.

[19] V. Girault and J.-L. Lions, *Two-grid finite-element schemes for the steady Navier-Stokes problem in polyhedra*, Portug. Math. **58** 1 (2001), pp. 25–57.

[20] V. Girault and P. A. Raviart, *Finite Element Methods for the Navier-Stokes Equations. Theory and Algorithms*, SCM **5**, Springer-Verlag, Berlin, 1986.

[21] V. Girault and L. R. Scott, *Finite-element discretizations of a two-dimensional grade-two fluid model*, M2AN **35**, (2002), pp. 1007–1053.

[22] V. Girault and L. R. Scott, *A quasi-local interpolation operator preserving the discrete divergence*, Calcolo **40**, (2003), pp. 1–19.

[23] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Pitman Monographs and Studies in Mathematics **24**, Pitman, Boston, MA, 1985.

[24] R. Haverkamp, *Eine Aussage zur* L^∞*-Stabilitat und zur genauen Konvergenzordnung der* H_0^1*-Projektionen*, Numer. Math. **44**, (1984), pp. 393–405.

[25] P. Hood and C. Taylor, *A numerical solution of the Navier-Stokes equations using the finite element technique*, Comp. and Fluids 1, (1973), pp. 73–100.

[26] R. B. Kellog and J. E. Osborn, *A regularity for the Stokes problem in a convex polygon*, J. Funct. Anal. **21**, (1976), pp. 397–431.

[27] J.-L. Lions and E. Magenes, *Problèmes aux Limites non Homogènes et Applications, I*, Dunod, Paris, 1968.

[28] F. Natterer, *Uber die punktweise Konvergenz Finiter Elemente*, Numer. Math. **25**, (1975), pp. 67–77.

[29] J. Nečas, *Les Méthodes Directes en Théorie des Équations Elliptiques*, Masson, Paris, 1967.

[30] J. A. Nitsche, *Lineare Spline-Funktionen und die Methoden von Ritz für elliptische Randwertprobleme*, Arch. Rat. Mech. Anal. **36**, (1970), pp. 348–355.

[31] J. A. Nitsche, L^∞*-Convergence of Finite Element Approximation*, Second Conf. on Finite Elements, Rennes, France, 1975.

[32] R. Rannacher and L. R. Scott, *Some optimal error estimates for linear finite element approximations*, Math. Comp. **38**, (1982), pp. 437–445.

[33] P. Saavedra and L. R. Scott, *Variational formulation of a model free-boundary problem*, Math. Comp. **57**, (1991), pp. 451–475.

[34] A. SCHATZ, *Pointwise error estimates and asymptotic error expansion inequalities for the finite element method on irregular grids*, Math. Comp. **67**, (1998), pp. 877–899.

[35] L. R. SCOTT, *Optimal L^∞ estimates for the finite element method on irregular meshes*, Math. Comp. **30**, (1976), pp. 681–697.

[36] L. R. SCOTT AND S. ZHANG, *Finite element interpolation of non-smooth functions satisfying boundary conditions*, Math. Comp. **54**, (1990), pp. 483–493.

[37] E. M. STEIN, *Note on singular integrals*, Proc. Amer. Math. Soc. **8**, (1957), pp. 250–254.

[38] R. STENBERG, *Analysis of finite element methods for the Stokes problem: a unified approach*, Math. Comp. **42**, (1984), pp. 9–23.